Kreiss Matrix Theorem

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1 Kreiss matrix theorem

For nonnormal matrix A, the behavior of $\|\exp(A)t\|$ can be quite complicated. Some commonly known results are:

$$\left\|e^{tA}\right\| \leq e^{\omega(A)t}, \quad \omega(A) := \sup(\sigma(\mathrm{Re}(A)), \tag{1}$$

$$\left\|e^{tA}\right\| \geq e^{\alpha(A)t}, \quad \alpha(A) := \lim_{t\to\infty} t^{-1}\log\left\|e^{tA}\right\|. \tag{2}$$

The *transient* effect is a very interesting numerical phenomenon. We consider the resolvent operator $(zI-A)^{-1}$ ($z\in\mathbb{C}$). If $\left\|(zI-A)^{-1}\right\|$ is very large (10^5) for some z with $\mathrm{Re}\, z = 10^{-2}$, then there must be a "transient growth" of magnitude at least 10^3. This leads to one half of the Kreiss matrix theorem. We define the *Kreiss constant* of A (w.r.t. the left half plane) as

$$\mathcal{K}(A) = \sup_{\mathrm{Re}\, z>0} (\mathrm{Re}\, z)\left\|(zI-A)^{-1}\right\|. \tag{3}$$

Then we have a sharper lower bound

$$\sup_{t\geq 0}\left\|e^{tA}\right\| \geq \mathcal{K}(A). \tag{4}$$

This result is known as the *Kreiss matrix theorem.*

Theorem 1. *For $A\in M_N(\mathbb{C})$, then we have*

$$\mathcal{K}(A) \leq \sup_{t\geq 0}\left\|e^{tA}\right\| \leq eN\mathcal{K}(A). \tag{5}$$

We can also consider the *Kreiss constant* w.r.t. the unit disk

$$\mathcal{K}^\circ(A) = \sup_{|z|>1} (|z|-1)\left\|(zI-A)^{-1}\right\|. \tag{6}$$

and yield the following *Kreiss matrix theorem* for $\mathcal{K}^\circ$:

Theorem 2. *For $A\in M_N(\mathbb{C})$, then we have*

$$\mathcal{K}^\circ(A) \leq \sup_{k\geq 0}\left\|A^k\right\| \leq eN\mathcal{K}^\circ(A). \tag{7}$$

We first prove Theorem 2.

Proof of Theorem 2. We first consider the left-hand inequality. Without lost of generality, we assume that A is power-bounded i.e. $\left\|A^k\right\| \leq C$ for some C and all k. We take the norm of the power series expansion

$$\left\|(zI-A)^{-1}\right\| = \left\|z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots\right\| \leq \frac{\sup_{k\geq 0}\left\|A^k\right\|}{|z|-1}. \tag{8}$$

Note that $|z|>1$ and A is power-bounded, therefore the spectrum of A necessarily lies in the closed unit disk $\overline{\mathbb{D}}$ and the series is guaranteed to converge. Therefore the left-hand inequality follows readily.

We now turn to the right hand part, note that

$$A^k = \frac{1}{2\pi\mathrm{i}}\int_G z^k (zI-A)^{-1}\mathrm{d}z. \tag{9}$$

Here, G is a closed contour enclosing the spectrum of A which itself lies in $\overline{\mathbb{D}}$ if $\mathcal{K}^\circ(A) < \infty$. Let u, v be arbitrary unit vectors, then

$$v^* A^k u = \frac{1}{2\pi \mathrm{i}} \int_G z^k R(z) \mathrm{d}z, \quad R(z) = v^*(zI - A)^{-1} u. \tag{10}$$

By integration by parts, we have

$$v^* A^k u = -\frac{1}{2\pi(k+1)\mathrm{i}} \int_G z^{k+1} R'(z) \mathrm{d}z. \tag{11}$$

We let $G = \{|z| = 1 + \frac{1}{k+1}\}$, then by the *Spijker's lemma* (The image, on the Riemann sphere, of any circle under a complex rational mapping, with numerator and denominator having degrees no more than n, has length no longer than $2n\pi$), we have

$$\left|v^* A^k u\right| \leq \frac{e}{2\pi(k+1)} \int_G |R'(z)| \mathrm{d}z \leq \frac{e}{2\pi(k+1)} 2\pi N \|R\|_\infty. \tag{12}$$

By definition, we have

$$\|R\|_\infty = (k+1) \sup_G \frac{1}{k+1} \left|v^*(zI - A)^{-1} u\right| \leq \sup_{|z|>1} (|z| - 1) \left\|(zI - A)^{-1}\right\| = \mathcal{K}^\circ(A). \tag{13}$$

Therefore,

$$\left\|A^k\right\| \leq eN\mathcal{K}^\circ(A). \tag{14}$$

□

The proof of Theorem 1 is essentially same as that of Theorem 2. In fact we have

Proof of Theorem 1. The left-hand inequality is trivial. For the right-hand inequality, we have

$$e^{tA} = \frac{1}{2\pi \mathrm{i}} \int_G e^{zt} (zI - A)^{-1} \mathrm{d}z. \tag{15}$$

Again, we define the (weak) resolvent function $R(z) = v^*(zI - A)^{-1} u$, then we have

$$v^* e^{tA} u = \frac{1}{2\pi \mathrm{i}} \int_G e^{zt} R(z) \mathrm{d}z = \frac{1}{2\pi t \mathrm{i}} \int_G e^{zt} R'(z) \mathrm{d}z. \tag{16}$$

We let $G = \mathrm{i}\mathbb{R} + 1/t$, then again by the *Spijker's lemma*

$$\left|v^* e^{tA} u\right| \leq \frac{e}{2\pi t} \int_G |R'(z)| \mathrm{d}z \leq \frac{e}{2\pi t} 2\pi N \|R\|_\infty = eN \sup_{z \in G} \frac{1}{t} \left|v^*(zI - A)^{-1} u\right| \leq eN \sup_{\mathrm{Re}\, z > 0} \mathrm{Re}\, z \left\|(zI - A)^{-1}\right\| = eN\mathcal{K}(A). \tag{17}$$

□

2 Restricted Kreiss constants

In fact, $\mathcal{K}(A)$ and $\mathcal{K}^\circ(A)$ can be strongly affected by some "defective" eigenvalues, such as pure imaginary eigenvalue for the former or unit imaginary eigenvalue for the latter. In these cases, the Kreiss constants can be infinite. In fact, this can be in some sense characterized by "restricted Kreiss constants".

Theorem 3. *We define*

$$\mathcal{K}_\delta(A) = \delta \sup_{z \in \mathrm{i}\mathbb{R} + \delta} \left\|(zI - A)^{-1}\right\|, \quad \mathcal{K}_\delta^\circ(A) = \delta \sup_{|z| = 1 + \delta} \left\|(zI - A)^{-1}\right\|. \tag{18}$$

Then, assume A has the Jordan decomposition $A = SJS^{-1}$ with S invertible, $d = \max_l d_l$ is the size of the largest Jordan block, we have

1. *If $\mathrm{Re}(\lambda) \notin (0, 2\delta)$ for any $\lambda \in \sigma(A)$ and $\delta \in (0, 1)$, then*

$$\mathcal{K}_\delta(A) \leq \kappa(S) \left(\frac{1}{\delta}\right)^{d-1} \frac{1 - \delta^d}{1 - \delta}. \tag{19}$$

2. *If* $|\lambda_l| \notin (1, 1+2\delta)$ *for any* $\lambda \in \sigma(A)$ *and* $\delta \in (0,1)$, *then*

$$\mathcal{K}_\delta^\circ(A) \leq \kappa(S) \left(\frac{1}{\delta}\right)^{d-1} \frac{1-\delta^d}{1-\delta}. \tag{20}$$

Proof. We claim, in both case ($z \in \mathrm{i}\mathbb{R} + \delta$ or $\in \{|z| = 1+\delta\}$,

$$\left\|(zI - A)^{-1}\right\| \leq \frac{\kappa(S)}{\delta^d} \cdot \frac{1-\delta^d}{1-\delta}. \tag{21}$$

Then the results follows readily by definition. In fact,

$$\left\|(zI - A)^{-1}\right\| \leq \|S\|\left\|S^{-1}\right\|\left\|(zI - J)^{-1}\right\| = \|S\|\left\|S^{-1}\right\|\left\|\mathrm{diag}(zI - J_l)^{-1}\right\| = \|S\|\left\|S^{-1}\right\| \max_l \left\|(zI - J_l)^{-1}\right\|. \tag{22}$$

Without loss of generalization, we assume that λ_l is the maximizer. Then we denote

$$N_l = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{d\times d}, \quad \|N_l\| = 1. \tag{23}$$

Then

$$\left\|(zI - J_l)^{-1}\right\| = \left\|[(z-\lambda_l)I - N_l]^{-1}\right\| \leq |z-\lambda_l|^{-1} \frac{1 - |z-\lambda_l|^{-d}\|N_l\|^d}{1 - |z-\lambda_l|^{-1}\|N_l\|} = \frac{1-\delta^{-d}}{\delta(1-\delta^{-1})}. \tag{24}$$

From which, we conclude our claim and thus prove the required results. ∎

References