MATH 206 Notes

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1 Hahn-Banach Theorem

Theorem 1.1 (Geometric Hahn-Banach for convex sets)**.** *V vector space over $\mathbb{K} = \mathbb{R}$. Suppose A is convex and linearly open. W (affined) subspace of V, with $A \cap W = \emptyset$, then $\exists$ a(n) (affined) hyperplane H containing W and disjoint from A.*

Remark 1.

- *We say $A \subset V$ is convex, if $\{t \in \mathbb{R} : x + ty \in A\}$ is an interval in $\mathbb{R}$ for any $x, y \in V$, or equivalently, for any $x, y \in A$ and $0 < \lambda < 1$, we have $\lambda x + (1 - \lambda)y \in A$.*
- *We say that $A \subset V$ is linearly open, if it is convex and for any $x, y \in A$, $\{t : x + ty \in A\}$ is an open interval in $\mathbb{R}$.*

Theorem 1.2 (Hahn-Banach Theorem for topological vector spaces)**.** *V topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose $A \subset V$ is convex and open, and W is a(n) (affined) subspace with $A \cap W = \emptyset$, then $\exists$ a(n) (affined)* ***closed*** *hyperplane H containing W and disjoint from A.*

Proof. (Sketch) A open implies A linearly open. Applying Theorem 1.1 there exists a hyperplane H containing W and disjoint from A. Since $V \setminus A$ is closed, we have $\overline{H} \subset V \setminus A$. $\operatorname{codim}\overline{H} \leq 1$ and $\overline{H}$ is not V, thus $\overline{H}$ is also a hyperplane. For $\mathbb{K} = \mathbb{C}$, we first view V as over $\mathbb{R}$ and find H_1 closed and $\operatorname{codim}_{\mathbb{R}} H_1 = 1$. Then consider $H = H_1 \cap (\mathrm{i}H_1)$. Then H is closed, a $\mathbb{C}$-subspace, and has codimension 2 over $\mathbb{R}$ thus codimension 1 over $\mathbb{C}$.

□

Theorem 1.3 (Hahn-Banach for LCTVS, closed version)**.** *V a locally convex topological vector space, B closed convex. $x \notin B$. Then there exists a continuous linear functional $f : V \to \mathbb{K}$ such that $\{f(x) = f(y)\} \cap B = \varnothing$. Or analytically,*

$$\inf_{y \in B} |f(x) - f(y)| > 0. \tag{1}$$

Proof. The strategy is recasting it to the open version. Find a balanced nbhd N of 0 such that $(x+N) \cap B = \varnothing$ and $x \notin A := B + N$. By the open version there exists closed H containing x and disjoint from A. This means $f(x + z_1) \neq f(y + z_2)$ for any $y \in B, z_1, z_2 \in N$, thus $f(N) \neq \{0\}$ but $f(N)$ is balanced around 0 in $\mathbb{K}$ thus $\inf_{y \in B} |f(x) - f(y)| > 0$. □

Corollary 1. *Same setting as above. Suppose $W \subset V$ a subspace, then $\overline{W}$ can be characterized as*

$$W = \bigcap_{\substack{f \text{ continuous linear functional} \\ f|_W = 0}} \ker f = \bigcap_{\substack{H \text{ closed hyperplane} \\ H \supset W}} H. \tag{2}$$

Proof. If $x \in \overline{W}$, $\ker f \supset W$ and by closedness we know that $\ker f \supset \overline{W}$ and we know that x in the intersection. Conversely, if $x \notin \overline{W}$, we want to find f continuous such that $f|_W = 0$ but $f(x) \neq 0$. This can be done by finding f such that $f(x) \neq f(y)$ for any $y \in \overline{W}$ ($\overline{W}$ closed and convex and away from x). If $f(y) \neq 0$, then $f(x) \neq f(f(x)/f(y) \cdot y) = f(x)$, which is a contradiction. □

Example 1 (Runge's approximation)**.** *Let $K \subset \mathbb{C}$ compact, suppose $\mathbb{C} \backslash K$ is connected. Then any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.*

Proof. We recall the Riesz representation of $C(K)^*$: for any linear functional L on $C(K)$ we can write it as

$$L(f) = \int_K f(z) \mathrm{d}\mu(z) \tag{3}$$

with some complex finite Borel measure μ supported on K. Now using the characterization of the closure of subspace, the closure of $\overline{W}$ with W the space of polynomials in $C(K)$ is

$$\overline{W} = \bigcap_{\substack{L \in C(K)^* \\ L|_W = 0}} \ker L. \tag{4}$$

Thus to show that any holomorphic f on nbhd(K) is in $\overline{W}$, we only need to show that for any μ with

$$\int_K z^n \mathrm{d}\mu(z) = 0, \forall n \geq 0, \tag{5}$$

we have

$$\int_K f(z) \mathrm{d}\mu(z) = 0. \tag{6}$$

This is done by the integral formula

$$\int_K f(z) \mathrm{d}\mu(z) = -\frac{1}{2\pi \mathrm{i}} \int \iint \frac{1}{\zeta - z} \bar{\partial} \varphi(\zeta) f(\zeta) \mathrm{d}\mu(z) \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\zeta, \tag{7}$$

and using the Taylor expansion and the moment condition to show that the integral is 0. □

Theorem 1.4 (Hahn-Banach for extension of functionals)**.** *Let V be a vector space (no topology at first). $p : V \to \mathbb{R}_{\geq 0}$ seminorm, $W \subset V$ subspace, $f : W \to \mathbb{K}$ linear functional with*

$$|f(w)| \leq p(w), \forall w \in W. \tag{8}$$

That is, f is a continuous linear functional on the locally convex TVS defined by the seminorm p. Then there exists an extension $\widetilde{f} : V \to \mathbb{K}$ of f to the whole V such that $\widetilde{f}$ is also a continuous linear functional with p, i.e.

$$\left|\widetilde{f}(v)\right| \leq p(v), \forall v \in V. \tag{9}$$

and $\widetilde{f}|_W = f$.

Proof. In the following discussion we view V as a LCTVS with the seminorm p. Define the convex open set $A = \{v \in V : p(v) < 1\}$ as well as the affined subspace $F = \{v \in W : f(v) = 1\}$. Then by $p \leq f$ we know that $A \cap F = \varnothing$. By geometry Hahn-Banach (open ver.) there exists a closed affined hyperplane $H = \{x : \widetilde{f}(x) = 1\}$ containing F and disjoint from A. Since $H \supset F$, we have for $\widetilde{f}(x) = 1$ for $x \in F$, thus $\widetilde{f}|_W = f$. Also, we can write $\left|\widetilde{f}(x)\right| = \left|\widetilde{f}(x)\right| \cdot \left|\widetilde{f}(x/\widetilde{f}(x))\right|$. Since $x/\widetilde{f}(x)$ is in H by definition, we have that $p\left(x/\widetilde{f}(x)\right) \geq 1$, thus $\left|\widetilde{f}(x)\right| \leq \left|\widetilde{f}(x)\right| p(x/\widetilde{f}(x)) = p(x)$. □

Example 2 (Existence of weak solution to linear PDEs in the Segal-Bergmann space)**.** *$V = L^2(\mathbb{R}^n, e^{-|x|^2/2}\mathrm{d}x)$, the Segal-Bergmann space. Consider the differential operator with constant coefficients $P(D)$ where P is a polynomial and $D = \frac{1}{\mathrm{i}}\partial$. We want to solve the equation $P(D)u = f$ for given $f \in V$, i.e. we want to find $u \in V$ such that the equation holds in the weak sense:*

$$\int uQ(D)v = \int fv, \forall v \in C_c^\infty(\mathbb{R}^n). \tag{10}$$

Here, $Q(D) = P(-D)$ is the formal adjoint of $P(D)$.

To do this, we consider the subspace $W = \{Q(D)v : v \in C_c^\infty(\mathbb{R}^n)\}$ of the space $U = L^2(\mathbb{R}^n, e^{|x|^2/2}\mathrm{d}x)$. We define a linear functional $L : W \to \mathbb{C}$ by

$$L(Q(D)v) = \int fv. \tag{11}$$

We show:

- *$\int |fv|^2 \leq \int |v|^2 e^{|x|^2} \int |f|^2 e^{-|x|^2} \lesssim \int |Q(D)v|^2 e^{|x|^2/2}$, thus L is well-defined and bounded.*
- *By Hahn-Banach, we can extend L to a bounded linear functional $\widetilde{L} : U \to \mathbb{C}$. By Riesz representation on L^2, there exists $u \in V$ such that $\widetilde{L}(g) = \int ug$ for any $g \in U$. Restrict it back to W, we get*

  $$\int uQ(D)v = \widetilde{L}(Q(D)v) = L(Q(D)v) = \int fv, \quad \forall v \in C_c^\infty(\mathbb{R}^n), \tag{12}$$

  which is exactly the weak solution we want.

2 "Great" Theorems in Functional Analysis

Theorem 2.1 (Baire)**.** *(E, d) complete metric space. $\{U_n\}_{n\geq 1}$ collection of open dense subsets of E. Then $\bigcap_{n\geq 1} U_n$ is dense in E. Equivalently, if $\{F_n\}_{n\geq 1}$ is a collection of closed subsets of E with empty interior, then $\bigcup_{n\geq 1} F_n$ has empty interior.*

Remark 2. *We say that the union of closed sets with empty interiors is a set of first category.*

Theorem 2.2 (Banach inverse mapping theorem)**.** *$T : F_1 \to F_2$ in an injective bounded linear operator between Banach (or Fréchet) spaces. We have the following dichotomy:* $\operatorname{Im} T$ *is either of first category in F_2, or* $\operatorname{Im} T = F_2$ *and T has a bounded inverse.*

Another version is: T is a bounded linear operator (not necessarily injective) between Banach (or Fréchet) spaces. Then $\operatorname{Im} T$ *is either of first category in F_2, or* $\operatorname{Im} T = F_2$. *This can be shown by applying the first version to the induced map $\widetilde{T} : F_1/\ker T \to F_2$.*

Proof. The proof is based on Baire's theorem. Suppose that $\operatorname{Im} T$ is not of first category. For fixed $r > 0$, denote

$$U = B_1(0, r) = \{x \in F_1 : \|x\| \leq r\}. \tag{1}$$

Then $F_1 = \bigcup_{n\geq 1} nU$. Thus

$$\operatorname{Im} T = T(F_1) \subset \bigcup_{n\geq 1} T(nU) = \bigcup_{n\geq 1} n\overline{T(U)}. \tag{2}$$

Note that this is a union of closed sets. Since we assumed that T is not of 1st cat., there exists some n such that there exists an open ball in F_2 contained in $n\overline{T(U)}$. By scaling, symmetry and convexity arguments, there exists some $\rho > 0$ such that $\overline{B_2(0, \rho)} \subset \overline{T(U)}$. Now we denote $U_k := 2^{-k} U$, then there exists $\rho_k > 0$ such that $\overline{B_2(0, \rho_k)} \subset \overline{T(U_k)}$. Thus we have the estimate $\rho_k \leq \|T\| 2^{-k} r$. Denote $V_k = B_2(0, \rho_k)$. We are in a position to show that T has a bounded inverse. For any $y \in V_1$, there exists $x_1 \in U_1$ such that $\|y - T x_1\| \leq \rho_2$, that is, there exists $y_1 \in V_2$ such that $y - T x_1 = y_1$. Repeating this process we get a sequence $\{x_k\}_{k\geq 1}$ with $x_k \in U_k$ such that

$$y - T\left(\sum_{k=1}^{N} x_k\right) = y_N \in V_{N+1}. \tag{3}$$

Since $\{\sum_{k=1}^{N} x_k\}_{N\geq 1}$ is a Cauchy sequence in F_1, it converges to some $x \in F_1$. Taking limit on both sides we get $Tx = y$. Also, we have the estimate

$$\|x\| \leq \sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k} r = r. \tag{4}$$

Since $y \in V_1$ is arbitrary, we have shown that for any $y \in B_2(0, \rho_1)$, there exists $x \in F_1$ with $\|x\| \leq r$ such that $Tx = y$. Thus the inverse is bounded by $\left\|T^{-1}\right\| \leq r/\rho_1$. □

Example 3. *$L^q([0, 1])$ is of first category in $L^p([0, 1])$ for $1 \leq p < q \leq \infty$. This is because the embedding is proper since $g(x) = x^{-\frac{1}{r}}$ with $p < r \leq q$ is in L^p but not in L^q. By the inverse mapping theorem, the image is of first category.*

Example 4 (The characteristic variety of a linear PDO)**.** *If for any $u \in C^m(\Omega)$, such that $P(D)u = 0$, we have $u \in C^{m+1}(\Omega)$, then an essential condition for this to hold is that the characteristic variety of P, defined as*

$$\Sigma = \{\xi \in \mathbb{C}^n : P(\xi) = 0\}, \tag{5}$$

satisfies $\lim_{|\xi|\to\infty,\xi\in\Sigma} |\mathrm{Im}\,\xi| = \infty$. *A consequence of this fact is that, for Schrödinger equation, there exists* C^2 *solution which is not* C^3.

Proof. Consider two Fréchet spaces $F_1 := \{u \in C^{m+1}(\Omega) : P(D)u = 0\}$ and $F_2 := \{u \in C^m(\Omega) : P(D)u = 0\}$. ($C^m$ space are Fréchet with the family of seminorms $\|u\|_K = \sum_{|\alpha|\le m} \sup_{x\in K} |\partial^\alpha u(x)|$ for $K \subset\subset \Omega$). This is an embedding. But we also hypothesize that this is a surjective, thus the inverse map is continuous. That is

$$\sum_{|\alpha|\le m+1} \sup_{x\in K} |\partial^\alpha u(x)| \lesssim \sum_{|\beta|\le m} \sup_{x\in K'} \left|\partial^\beta u(x)\right|, \forall u \in F_1 = F_2. \tag{6}$$

We take the characteristic function $u(x) = e^{i\xi\cdot x}$ with $\xi \in \Sigma$. Then $P(D)u = P(\xi)e^{i\xi\cdot x} = 0$. Plugging in we get $\frac{A_{m+1}(\zeta)}{A_m(\zeta)} \lesssim \sup_{x\in K, x'\in K'} e^{-\mathrm{Im}\langle\zeta, x-x'\rangle} \lesssim e^{|\mathrm{Im}\,\zeta|}$ for any $\zeta \in \Sigma$, where $A_k(\zeta) = \sum_{|\alpha|\le k} |\zeta^\alpha|$. Since $\lim_{|\xi|\to\infty,\xi\in\Sigma} \frac{A_{m+1}(\xi)}{A_m(\xi)} = \infty$, we must have $\lim_{|\xi|\to\infty,\xi\in\Sigma} |\mathrm{Im}\,\xi| = \infty$. □

Theorem 2.3 (Closed graph theorem)**.** *$T : \mathcal{D}(T) \to F_2$ linear (can be unbounded* a priori*) and has closed graph ($\mathcal{D}(T) \subset F_1$, F_1, F_2 Banach or Fréchet spaces). Then either $\mathcal{D}(T)$ is of first category in F_1, or $\mathcal{D}(T)F_1 = F_1$ and T is bounded.*

Proof. Consider the continuous projectors π_1 and π_2. Since $\mathcal{G}(T)$ is closed, $\pi_1|_{\mathcal{G}(T)}$ is also continous and $\mathrm{Im}\,\pi_1|_{\mathcal{G}(T)} = \mathcal{D}(T)$. If $\mathcal{D}(T)$ is not of 1st cat., then $\mathcal{D}(T) = F_1$. In the latter case the inverse $[\pi_1|_{\mathcal{G}(T)}]^{-1} : F_1 \to \mathcal{G}(T)$ is bounded. Thus $T = \pi_2 \circ [\pi_1|_{\mathcal{G}(T)}]^{-1}$ is also bounded. □

Theorem 2.4 (Banach-Steinhaus uniform boundedness principle)**.** *F Fréchet, V LCVS, Φ a family of linear conti. operators $F \to V$. Define the set of pts whose orbits under Φ are bounded: $\Sigma = \{x \in F : \Phi x$ is bounded in $V\}$. Then either Σ is of 1st cat., or $\Sigma = F$ and Φ is equi-continuous.*

Proof. Fixed arbitrary balanced nbhd U of 0 in V. We write $\Sigma = \bigcup_{n\ge1} nA(U)$, where $A(U)$ is the intersection of the preimages of U in V under all maps T in Φ (by the definition of boundedness). If Σ is not of 1st cat., then there exists some n such that $nA(U)$ contains a balanced nbhd W of 0 in F, i.e. $T(W) \subset U$. □

Example 5 (Divergence of Fourier series)**.** *There exists a continuous function on the circle whose Fourier series diverges at a point. Since we can choose f such that*

$$|S_N f(0)| \gtrsim L_N \|f\|_\infty, \quad L_N = \|D_N\|_1 \gtrsim \sum_{k=1}^{N} \frac{1}{k} \sim O(\log N), \tag{7}$$

where D_N is the Dirichlet kernel. Then the operators $T_N : C(\mathbb{T}) \to \mathbb{C}$ defined by $T_N f := S_N f(0)$ is not equi-continuous, thus there must be some $\varphi \in C(\mathbb{T})$ such that $\sup_N |S_N \varphi(0)| = \infty$.

3 Fredholm Theory

Theorem 3.1 (Algebraic Fredholm theory)**.** *Let $T : V_1 \to V_2$ such that* $\dim\ker T = n_+ < \infty$ *and* $\dim \mathrm{coker} T = n_- < \infty$. *Then there exists $R_- : \mathbb{K}^{n_-} \to V_2$ injective and $R_+ : V_1 \to \mathbb{K}^{n_+}$ surjective such that the operator*

$$\begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix} : V_1 \oplus \mathbb{K}^{n_-} \to V_2 \oplus \mathbb{K}^{n_+} \tag{1}$$

is an isomorphism.

Remark 3. *Briefly, we can always "complete" a Fredholm operator to an isomorphism by adding finite-dimensional spaces.*

Proof. The proof is based on construction.

- We take the basis $\{x_1, \cdots, x_{n_+}\}$ of $\ker T$, by Hahn-Banach there exists a "dual basis" of linear functionals on V_1, $\{x_1^*, \cdots, x_{n_+}^*\}$ such that $\langle x_i^*, x_j \rangle = \delta_{ij}$. We define $R_+ : V_1 \to \mathbb{K}^{n_+}$ as $u \mapsto (\langle x_1^*, u\rangle, \cdots, \langle x_{n_+}^*, u\rangle)$. This is surjective by construction. Moreover, $\ker T \cap \ker R_+ = \{0\}$.
- We take the basis $\{[y_1], \cdots, [y_{n_-}]\}$ of $\operatorname{coker} T = V_2 / \operatorname{Im} T$. We define $R_- : \mathbb{K}^{n_-} \to V_2$ as $R_-(e_i) = y_i$, where $\{e_i\}$ is the standard basis of $\mathbb{K}^{n_-}$. This is injective by construction. Moreover, $\operatorname{Im} T \cap \operatorname{Im} R_- = \{0\}$.
- We check $\widetilde{T} := \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix}$ is (1) surjective since $\widetilde{T}\left(\begin{pmatrix} u \\ \begin{pmatrix} c_1 \\ \vdots \\ c_{n_-} \end{pmatrix} \end{pmatrix}\right) = \begin{pmatrix} Tu + \sum_{i=1}^{n_-} c_i y_i \\ R_+ u \end{pmatrix}$, and R_+ is surjective.

  (2) injective since $\widetilde{T}\begin{pmatrix} u \\ u_- \end{pmatrix} = 0$ iff $Tu + R_- u_- = 0$ and $R_+ u = 0$. Note that $Tu \in \operatorname{Im} T \cap \operatorname{Im} R_- = \{0\}$, and thus $Tu = R_- u_- = 0$. Since R_- is injective, we have $u_- = 0$ and thus $u \in \ker T \cap \ker R_+ = \{0\}$, thus $u = 0$.

□

Proposition 1 (Schur complement formula)**.** $\widetilde{T} = \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix}$ *is invertible with* $\widetilde{T}^{-1} = \begin{pmatrix} E & E_- \\ E_+ & E_{-+} \end{pmatrix}$*. Then* $\dim \ker T = \dim \ker E_{-+}$, $\dim \operatorname{coker} T = \dim \operatorname{coker} E_{-+}$. *In particular, T^{-1} exists iff E_{-+} is invertible, and in this case*

$$T^{-1} = E - E_- E_{-+}^{-1} E_+. \tag{2}$$

Note that in this case $\operatorname{Ind} T = \dim \ker T - \dim \operatorname{coker} T = 0$*. Therefore E_{-+} is a $n_\pm \times n_\pm$ matrix ($n_+ = n_-$).*

Theorem 3.2. *$T : B_1 \to B_2$ Fredholm operator between Banach spaces, $S : B_1 \to B_2$ continuous with $\|S\| \ll 1$, then*

- *$T + S$ is also Fredholm,*
- $\operatorname{Ind}(T + S) = \operatorname{Ind}(T)$,
- $\dim \ker(T + S) \leq \dim \ker T$, $\dim \operatorname{coker}(T + S) \leq \dim \operatorname{coker} T$.

Proof. $\|S\| \ll 1$ implies that $\left\| \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \right\| \ll 1$. Thus $\widetilde{T} + \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ is also invertible. Denote its inverse as $\begin{pmatrix} \widetilde{E}^S & \widetilde{E}_-^S \\ \widetilde{E}_+^S & \widetilde{E}_{-+}^S \end{pmatrix}$. Thus $\dim \ker(T + S) = \dim \ker E_{-+}^S \leq n_+$, $\dim \operatorname{coker}(T + S) = \dim \operatorname{coker} E_{-+}^S \leq n_-$, thus $T + S$ is Fredholm with $\operatorname{Ind}(T + S) = n_+ - n_- = \operatorname{Ind} T$. □

Followings are some basic facts about compact operators:

Proposition 2. • *$\mathcal{L}_c(B_1, B_2) \subset \mathcal{L}(B_1, B_2)$ is a closed two-sided ideal.*

- *If T is semi-Fredholm i.e.* $\dim\ker T < \infty$ *and* $\operatorname{Im} T$ *is closed, K is a compact operator, then $T + K$ is also semi-Fredholm with* $\operatorname{Ind}(T + K) = \operatorname{Ind} T$.

Theorem 3.3 (Atkinson characterization of Fredholm operators)**.** *$T : B_1 \to B_2$ bounded linear operator between Banach spaces, then TFAE:*

- *T is Fredholm,*
- *$\exists E : B_2 \to B_1$, such that $TE = I + R_1$, $ET = I + R_2$ with R_1, R_2 finite rank operators,*
- *$\exists E : B_2 \to B_1$, such that $TE = I + K_1$, $ET = I + K_2$ with K_1, K_2 compact operators.*
- *$\exists E_1 : B_2 \to B_1$ and $E_2 : B_2 \to B_1$ such that $TE_1 = I + K_1$, $E_2T = I + K_2$ with K_1, K_2 compact operators.*

Remark 4. *Briefly, Fredholm operators are invertible up to compact perturbations.*

Proof. (1) $\Longrightarrow$ (2) By algebraic Fredholm theory,

$$\begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix}\begin{pmatrix} E & E_- \\ E_+ & E_{-+} \end{pmatrix} = \begin{pmatrix} TE + R_-E_- & * \\ * & R_+E_+ \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3}$$

$$\begin{pmatrix} E & E_- \\ E_+ & E_{-+} \end{pmatrix}\begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} ET + E_-R_+ & * \\ * & E_+R_- \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4}$$

Thus reading the first row/column we get the desired result with $R_1 = R_-E_-$ and $R_2 = E_-R_+$, both finite rank.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are trivial.

(4) $\Longrightarrow$ (1) $TE_1 = I + K_1$, thus $\operatorname{Im} T \supset \operatorname{Im}(I + K_1)$, which implies $\operatorname{codim} \operatorname{Im} T < \infty$ (since $I + K_1$ is Fredholm with index 0, because K_1 is compact). Similarly, $E_2T = I + K_2$ implies $\dim\ker T < \infty$. Thus T is Fredholm. $\square$

Example 6 (Application: Toeplitz operators)**.** *$H_2 = \{f \in L^2 : f(x) = \sum_{n\geq 0} a_n e^{\mathrm{i}nx}\}$ the Hardy space on the circle. Let $f \in C(\mathbb{T})$ and $f(e^{\mathrm{i}\theta}) = \sum_{m\in\mathbb{Z}} \hat{f}_m e^{\mathrm{i}m\theta}$ be its Fourier series. The Toeplitz operator $T_f : H_2 \to H_2$ is defined as $T_f u = P(fu)$ where P is a projection from L^2 to H_2 (view as a strip-like matrix acting on ℓ^2 the Fourier coefficients). Then T_f is Fredholm with*

$$\operatorname{Ind} T_f = -\textit{winding number of } f \textit{ around } 0 = -\frac{1}{2\pi}[\arg f(e^{\mathrm{i}\theta})]_{\theta=0}^{2\pi} = -\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{f'(e^{\mathrm{i}\theta})}{f(e^{\mathrm{i}\theta})}\mathrm{d}\theta. \tag{5}$$

Here $|f| > 0$ on $\mathbb{T}$.

Proof. We have $f, g \in C(\mathbb{T})$ implies $T_fT_g - T_{fg} \in \mathcal{L}_c(H_2)$. This is because for p, q being trigonometric polynomials we have by suitable cutoffs that $T_pT_q - T_{pq}$ is of finite rank. Thus by the fact that $\mathcal{L}$ is compact and that $\left\|T_f\right\| = \|f\|_\infty$ and that trigonometric polynomials are dense in $C(\mathbb{T})$, we get the desired result. Now if f is nowhere vanishing, then there exists $g \in C(\mathbb{T})$ such that $fg = 1$. Thus $T_fT_g = I + K_1$, $T_gT_f = I + K_2$ with K_1, K_2 compact. By Atkinson's theorem, T_f is Fredholm. We compute that $\operatorname{Ind} T_{e_n}$ which is a translation operator is $-n$. Thus $\operatorname{Ind} T_{e_n} = -$winding number of e_n around 0. Now it remains to show that for f with zero argument variation, $\operatorname{Ind} T_f = 0$. In such case we write $f = e^F$ with $F \in C(\mathbb{T})$ continuous. Note that index is stable on a homotopy we have $\operatorname{Ind} T_f = \operatorname{Ind} T_{e^{tF}}|_{t=0}^{1} = \operatorname{Ind} T_1 = 0$. $\square$

Theorem 3.4 (Riesz)**.** *Suppose $S \in \mathcal{L}_c(B)$, $N_k := \ker(I+S)^k$, then there exists some J such that $N = \bigcup_{k\geq 0} N_k = \ker(I+S)^J$. Moreover, for the same J, $F = \bigcap_{j\geq 0} F_j = \mathrm{Im}(I+S)^J$ with $F_j = \mathrm{Im}(I+S)^j$, and we have the topological direct sum decomposition $B = N \oplus F$, both N and F are invariant under S, and $(I+S)|_F$ is invertible.*

Proof. If the ascending chain N_k never stabilizes, then by the proof of "identity is compact iff finite dimensional" we can find $x_k \in N_k$ such that $\|x - x_k\| \geq 1$ for any $x \in N_{k-1}$ and $\|x_k\| = 1$. Let $x = (I+S)x_k - Sx_j$ ($j \leq k-1$), then since N_{k-1} is S-invariant, we have that $x \in N_{k-1}$ and thus $\|x - x_k\| = \left\|S(x_k - x_j)\right\| \geq 1$. This means that $\{Sx_k\}$ has no Cauchy subsequence, contradicting the compactness of S. Thus N_k stabilizes at some J. Similarly for F_j by the fact that $\dim\ker(I+S)^{J+1} = \dim\ker(I+S)^J$ implies $\dim\mathrm{coker}(I+S)^{J+1} = \dim\mathrm{coker}(I+S)^J$ implies $\mathrm{Im}(I+S)^{J+1} = \mathrm{Im}(I+S)^J$. (Note that S is compact and thus $I+S$ is Fredholm with index $\mathrm{Ind}(I+S) = \mathrm{Ind}(I) = 0$.)

Now N is a finite union of finite dimensional spaces, thus is finite dimensional (or directly note that $(I+S)^J = I +$ compact is Fredholm), and F is closed, thus this direct sum is topological. □

Theorem 3.5 (Analytical Fredholm theory)**.** *We assume that $\Omega \ni z \mapsto A(z)$ is a meromorphic family of Fredholm operators on B, such that $A(z_0)^{-1}$ exists for some z_0. Then $\Omega \ni z \mapsto A(z)^{-1}$ is meromorphic with values in $\mathcal{L}(B)$, with poles of finite rank.*

Proof. Since $\mathrm{Ind}(A(z_0)) = 0$ and index is stable under continuous perturbations, we have $\mathrm{Ind}(A(w)) = 0$ for any $w \in \Omega$ (Ω is a connected open set). Thus the Grushin opeartor $\begin{pmatrix} A(w) & R_-^w \\ R_+^w & 0 \end{pmatrix}$ is invertible. In a neighborhood of w, $z \mapsto \begin{pmatrix} A(z) & R_-^w \\ R_+^w & 0 \end{pmatrix}$ is still invertible. Since $z \mapsto A(z)$ is holomorphic, we have $z \mapsto E_*^w(z)$ ($* =$ NONE, $-$, $+$, $-+$) are also holomorphic. Thus by Schur complement formula, we have

$$A(z)^{-1} = E^w(z) - E_-^w(z)(E_{-+}^w(z))^{-1}E_+^w(z). \tag{6}$$

By finite-dimensional argument, since $\det E_{-+}^w(z)$ is holomorphic and non-vanishing at $z = w$, $E_-^w(z)(E_{-+}^w(z))^{-1}E_+^w(z)$ is meromorphic with poles of finite rank at $z = w$. Together with the holomorphy of $E^w(z)$, we get the desired result. □

Example 7 (Application: Riesz projectors)**.** *$P : X_1 \to X_2$ is continuous inclusion, then by analytic Fredholm theory,*

$$(P - zI)^{-1} = -\left(\frac{\Pi}{z - z_1} + \cdots + \frac{(P - z_1)^{N+1}\Pi}{(z - z_1)^N}\right) + R_0(z) \tag{7}$$

where z_1 is a pole, $z \mapsto R_0(z)$ is holomorphic, and $\Pi : X_2 \to \ker(P - z_1 I)^N$ is the Riesz projector onto the generalized eigenspace of P at z_1. We can recover the Atkinson result by noting $N = \Pi X_1$, $F = (I - \Pi)X_1$.

Proof. WLOG $z_1 = 0$, applying analytic Fredholm theory to $P - zI$, we have

$$(P - zI)^{-1} = \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_N}{z^N} + R_0(z), \tag{8}$$

$$\Pi := -A_1 = -\frac{1}{2\pi \mathrm{i}} \oint_{|z|=\epsilon} (P - zI)^{-1} \mathrm{d}z, \tag{9}$$

$$\Pi^2 = -\frac{1}{4\pi} \oint_{|z|=\epsilon} \oint_{|w|=\epsilon} (P - zI)^{-1}(P - wI)^{-1} \mathrm{d}z\mathrm{d}w = -\frac{1}{2\pi \mathrm{i}} \oint_{|z|=\epsilon} (P - zI)^{-1} \mathrm{d}z = \Pi, \tag{10}$$

applying $(P-z)$ on both sides and matching the terms, we get $PA_{-N}=0$ and $PA_{-k}=A_{-k-1}(k<N)$, and

$$\Pi+(P-z)R(z)=I\Rightarrow\quad (P-z)R(z)=I-\Pi\Rightarrow(P-z)(I-\Pi)R(z)=I-\Pi\Rightarrow PR(0)(I-\Pi)=I-\Pi,\quad (11)$$

therefore $P|_{\mathrm{Im}(I-\Pi)}$ is invertible, and

$$N=\mathrm{Im}\,\Pi=\ker P^N,\quad F=\mathrm{Im}(I-\Pi)=\mathrm{Im}\,P^N. \quad (12)$$

We recover the Atkinson decomposition $X_1=N\oplus F$. □

4 Duality

First we give the two main results for dual operators.

Theorem 4.1. *$T\in\mathcal{L}(B_1,B_2)$, $\mathrm{Im}\,T$ closed, then $\mathrm{Im}\,T^*$ is also closed and*

- $(\ker T)^\circ=\mathrm{Im}\,T^*$, $\ker T^*=(\mathrm{Im}\,T)^\circ$,
- $\dim\ker T=\dim\mathrm{coker}T^*$, $\dim\mathrm{coker}T=\dim\ker T^*$,
- *In particular, if T is Fredholm, then so is T^* with* $\mathrm{Ind}T^*=-\mathrm{Ind}T$.

The proof is based on the following algebraic result:

Theorem 4.2. *$W\subset B$ is closed, define*

$$\iota^*:B^*\to W^*,\quad \iota^*|_{W^\circ}=0,\quad (\iota^*)':B^*/W^\circ\to W^*\text{ is an isometric isomorphism.} \quad (1)$$

$$q^*:(B/W)^*\to B^*,\quad q^*((B/W)^*)=W^\circ,\quad q^*:(B/W)^*\to W^\circ\text{ is an isometric isomorphism.} \quad (2)$$

Proof.
- $\iota^*\xi=\xi|_W$ since ι is an embedding. Thus $\iota^*\xi=0$, if and only if $\xi\in W^\circ$. For any $\eta\in W^*$, by HB $\exists\,\xi\in B^*$ such that $\xi|_W=\eta$ and $\|\xi\|=\|\eta\|$. Thus $\|\iota^*\xi\|=\|\eta\|=\|[\xi]_{W^\circ}\|$. Thus $(\iota^*)'$ is an isometric isomorphism.
- For any $\zeta\in(B/W)^*$, we consider

$$\langle x,q^*\zeta\rangle=\langle qx,\zeta\rangle=\langle[x]_W,\zeta\rangle \quad (3)$$

Thus $q^*\zeta:x\mapsto\langle[x]_W,\zeta\rangle$, therefore $q^*\zeta\in W^\circ$ because $[x]_W=[0]_W$ for $x\in W$. We take any $\xi\in W^\circ$, $\xi':B/W\to\mathbb{K}$ is well-defined since $\xi|_W=0$. Moreover, $\|\xi'\|=\|\xi\|$. Therefore q^* is an isometric isomorphism.

□

Proof of the theorem. If T is bijective, then there exists S such that $ST=I_{B_1}$, $T^*S^*=I_{B_1^*}$, $S^*T^*=I_{B_2^*}$, thus all the results hold trivially.

If T is not bijective, define

$$T_1:B_1\to B_1/\ker T,\quad T_2:B_1/\ker T\to\mathrm{Im}\,T,\quad T_3:\mathrm{Im}\,T\to B_2, \quad (4)$$

then T_1 is surjective, T_2 is bijective, T_3 is injective. $T=T_3\circ T_2\circ T_1$, thus $T^*=T_1^*\circ T_2^*\circ T_3^*$.

$$T_1^*:(B_1/\ker T)^*\to B_1,\quad T_3^*:B_2^*\to(\mathrm{Im}\,T)^*. \quad (5)$$

Since $(B_1/\ker T)^* \cong (\ker T)^\circ$, thus $\operatorname{Im} T_1^* \cong (\ker T)^\circ$. Since $(\operatorname{Im} T)^* \cong B_2^*/(\operatorname{Im} T)^\circ$, thus $\ker T_3^* = (\operatorname{Im} T)^\circ$.

T_1 surjective $\implies$ T_1^* injective, T_3 injective $\implies$ T_3^* surjective. T_2 bijective $\implies$ T_2^* bijective. Thus we have

$$\operatorname{Im} T^* = \operatorname{Im}(T_1^* T_2^* T_3^*) = \operatorname{Im} T_1^* \cong (\ker T)^\circ, \tag{6}$$

$$\ker T^* = \ker(T_1^* T_2^* T_3^*) = \ker T_3^* = (\operatorname{Im} T)^\circ. \tag{7}$$

For the dimensional relations, we just note that the dimension of a subspace equals to its dual space, thus

$$\dim \ker T = \dim(\ker T)^* = \dim(B_1^*/(\ker T)^\circ) = \dim(B_1^*/\operatorname{Im} T^*) = \dim \operatorname{coker} T^*, \tag{8}$$

$$\dim \operatorname{coker} T = \dim(\operatorname{coker} T)^* = \dim(B/\operatorname{Im} T)^* = \dim(\operatorname{Im} T)^\circ = \dim \ker T^*. \tag{9}$$

□

Theorem 4.3. *If $T \in \mathcal{L}_c(B_1, B_2)$, then $T^* \in \mathcal{L}_c(B_2^*, B_1^*)$.*

Proof. We need to show that for $\xi_n \in B_2^*$, $T^*\xi_n$ has a convergent subsequence. We define $K = \{Tx : \|x\|_{B_1} \leq 1\} \subset\subset B_2$. Note that $\{\xi_n\}$ is in $C(K)$ and $|\xi_j y| \leq \|\xi_j\| \|y\| = \|y\|$. Therefore by Arzelà-Ascoli theorem, there exists a convergent subsequence $\xi_{n_k} \to \xi$ in $C(K)$. By passing to a subsequence we assume that ξ_j is convergent in $C(K)$, thus

$$\left\|T^*\xi_j - T^*\xi_k\right\|_{B_1^*} \leq \sup_{x \in B_1, \|x\| \leq 1} \left|\langle T^*(\xi_j - \xi_k), x\rangle\right| = \sup_{x \in B_1, \|x\| \leq 1} \langle \xi_j - \xi_k, Tx\rangle \leq \left\|\xi_j - \xi_k\right\|_{C(K)} \to 0. \tag{10}$$

□

Theorem 4.4 (Banach-Alaoglu)**.** *U is the closed unit ball in B^*, then U is compact in the weak-$*$ topology.*

Proof. Only prove for the case B is separable. Let $\{x_n\}$ be a dense subset of B. We consider the seminorms

$$p_n(\xi) = |\langle \xi, x_n\rangle|, \quad \xi \in B^*. \tag{11}$$

We claim that the topology generated by these seminorms is exactly the weak-$*$ topology on B^*. By the following lemma, indeed, enough to show that for any given $x, \varepsilon > 0$, there exists some $j, \delta > 0$ such that $\{\xi : |\langle x, \xi\rangle| < \varepsilon\} \supset \{\xi : |\langle x_j, \xi\rangle| < \delta\}$. We find x_j such that $\|x - x_j\| \leq \frac{\varepsilon}{10}$ and take $\delta = \frac{\varepsilon}{10}$, we have

$$|\langle x, \xi\rangle| \leq \left\|x - x_j\right\| \|\xi\| + \delta \leq \rho + \delta < \varepsilon. \tag{12}$$

Thus we only need to show that $\xi_n \in U$ has a subsequence such that $\langle x_j, \xi_{n_k}\rangle$ converges for any j. By diagonal argument, we can find such subsequence. □

Example 8. *$\{f_n\}$ a bounded sequence in L^q. By Banach Alaoglu, there exists a subsequence $\{f_{n_k}\}$ and some $f \in L^q$ such that for any $g \in L^p$ ($\frac{1}{p} + \frac{1}{q} = 1$), we have*

$$\int f_{n_k} g \mathrm{d}x \to \int f g \mathrm{d}x. \tag{13}$$

Lemma 1. *$L : F \to \mathbb{K}$ is continuous linear functional with respect to the topology $\sigma(F, G)$ (geneerated by the huge number of seminorms $x \mapsto |\langle x, y\rangle|$ for each fixed $y \in G$, and $y \mapsto \langle x, y\rangle$ is linear for each fixed $x \in F$). Then $\exists y \in G$ such that $L(x) = \langle x, y\rangle$ for any $x \in F$. In particular, if $\xi_n - \xi_m$ converges to 0 in weak-$*$, then there exists some $\xi \in B^*$ such that ξ_n converges to ξ in weak-$*$.*

Proof. Since L is continuous, there exists some finite number of seminorms and $C > 0$ such that

$$|L(x)| \leq C \sum_{j=1}^{N} \left|\langle x, y_j \rangle\right|, \quad \forall x \in F. \tag{14}$$

We define $N = \{x \in F : \langle x, y_j \rangle = 0, j = 1, \cdots, N\}$, then $L|_N = 0$, F/N is finite dimensional. We define L' and Y_j on F/N as

$$L'([x]) = L(x), \quad Y_j([x]) = \langle x, y_j \rangle. \tag{15}$$

Then they are all well-defined. Note that

$$\dim \operatorname{Span}(Y_1, \cdots, Y_N) \leq N = \dim(F/N)^* - \dim[\operatorname{Span}(Y_1, \cdots, Y_N)]^\circ = \dim(F/N) - \dim\{[0]\} = \dim(F/N), \tag{16}$$

thus we have $F/N = \operatorname{Span}(Y_1, \cdots, Y_N)$. Therefore there exists c_j such that $L' = \sum_{j=1}^N c_j Y_j$. Thus for any $x \in F$, we have

$$L(x) = L'([x]) = \sum_{j=1}^{N} c_j Y_j([x]) = \sum_{j=1}^{N} c_j \langle x, y_j \rangle = \langle x, \sum_{j=1}^{N} c_j y_j \rangle. \tag{17}$$

We take $y = \sum_{j=1}^N c_j y_j \in G$, and we are done. □

Corollary 2. *$M \subset B^*$, then M is closed in weak-$*$ iff $M \cap U$ is closed in weak-$*$.*

Corollary 3. *If $\operatorname{Im} T^*$ is closed, then $\operatorname{Im} T$ is also closed.*

5 Spectral Theorem for Self-Adjoint Operators

Example 9. *$Tu = \frac{1}{i}u'$ on $H = L^2(0, 1)$. If $\mathcal{D}_1 = C_c^\infty(0, 1)$, then T is symmetric (since no boundary terms). If $\mathcal{D}_2 = \{u \in L^2 : u' \in L^2\}$ (u' is the weak derivative), then T is not symmetric.*

Definition 5.1. *T densely defined, we say that $u \in \mathcal{D}_{A^*}$ if $v \mapsto \langle u, Av \rangle$ is continuous on $\mathcal{D}_A$ with respect to the norm of H. In such case, since $\mathcal{D}_A$ is dense in H, the continuous linear functional uniquely extends to H and thus by Riesz representation theorem there exists a unique $f \in H$ such that $\langle u, Av \rangle = \langle f, v \rangle$ for any $v \in \mathcal{D}_A$. We define $A^*u = f$.*

Proposition 3. *A closed and symmetric, then for any $z \in \mathbb{C} \setminus \mathbb{R}$, we have*

- $\ker(A - zI) = \{0\}$,
- $\operatorname{Im}(A - zI)$ *is closed,*
- $|\operatorname{Im} z| \|u\| \leq \|(A - zI)u\|$ *for any* $u \in \mathcal{D}_A$.

Proof. We only need to prove the last item, since the first two follow directly. We compute

$$\|(A - z)u\|^2 = \|Bu\|^2 + (\operatorname{Im} z)^2 \|u\|^2 \geq (\operatorname{Im} z)^2 \|u\|^2. \tag{1}$$

□

Proposition 4. *A symmetric and densely defined, then define*

$$n_{\pm} = \dim\ker(A^* \pm \mathrm{i}) = \dim\operatorname{coker}(A \mp \mathrm{i}). \tag{2}$$

We have that $\operatorname{codim}\operatorname{Im}(A - z) = \dim\ker(A^* - \overline{z})$ *is constant on* $\{\operatorname{Im} z > 0\}$ *and* $\{\operatorname{Im} z < 0\}$ *respectively, equal to* n_+ *and* n_-. *(They are called the deficiency indices of A.)*

Proof. We first directly verify that $\ker(A^* - \overline{z}) = [\operatorname{Im}(A - z)]^{\perp}$ by the density of $\mathcal{D}_A$. Thus $\operatorname{codim}\operatorname{Im}(A - z) = \dim\operatorname{coker}(A^* - \overline{z})$. Now we want to apply the Fredholm theory argument but cannot directly do this for unbdd operators. Thus we consider the graph operators:

$$T : \mathcal{G}(A) \ni (u, Au) \mapsto (A - z)u \in H, \tag{3}$$

$$S : \mathcal{G}(A) \ni (u, Au) \mapsto -\zeta u \in H. \tag{4}$$

For $\operatorname{Im} z > 0$ and $|\zeta| \ll 1$, note that $\ker(A - z) = 0$ and $\operatorname{Im}(A - z)$ is closed ($z \notin \mathbb{R}$), we have $\ker T = 0$ and $\operatorname{Im} T$ is closed (semi-Fredholm). By the Fredholm theory we have $\operatorname{Ind}(T + S) = \operatorname{Ind}(T)$ for ζ small enough. That is, $\operatorname{Ind} T = -\operatorname{codim}(\operatorname{Im}(A - z)) = -\operatorname{codim}(\operatorname{Im}(A - z - \zeta))$. Thus $\operatorname{codim}\operatorname{Im}(A - z)$ is constant for $\operatorname{Im} z > 0$. Similarly for $\operatorname{Im} z < 0$. □

Proposition 5. *A is self-adjoint iff* $n_+ = n_- = 0$.

Proof. If A is self-adjoint, then $n_{\pm} = \dim\ker(A^* \pm \mathrm{i}) = \dim\ker(A \pm \mathrm{i}) = 0$, since $\pm\mathrm{i} \notin \mathbb{R}$ and the fact that A is closed and symmetric (self-adjointness implies closedness, since an adjoint opeartor is always closed). If $u \in \mathcal{D}_{A^*}$ and $\operatorname{Im} z > 0$, since $n_+ = 0$, we have $\operatorname{Im}(A - z) = H$, thus $\exists v \in \mathcal{D}_A$, $(A - z)v = (A^* - z)u$, since A is symmetric, we have $(A^* - z)(u - v) = 0$, by $n_- = 0$ we have $u = v \in \mathcal{D}_A$. □

Theorem 5.2 (Kato)**.** *Suppose A is s.a. with domain* $\mathcal{D}_A$, *and V is symmetric with* $\mathcal{D}_V \supset \mathcal{D}_A$. *If there exists* $0 \leq a < 1$, $b > 0$ *such that for any* $u \in \mathcal{D}_A$, $\|Vu\| \leq a\|Au\| + b\|u\|$, *then* $A + V$ *with domain* $\mathcal{D}_A$ *is also s.a.*

Proof. By the assumption graph norms of A and the norm of $A + tV$ $(0 \leq t \leq 1)$ are equivalent. We WTS $n_{\pm}(t) := \operatorname{codim}(A + tV \pm \mathrm{i})$. Once we have this, then $n_{\pm}(0) = 0$ implies $n_{\pm}(1) = 0$, thus $A + V$ is s.a. We consider the graph operators:

$$T : \mathcal{G}(A + tV \pm \mathrm{i}) \to H, \quad T(u, (A + tV \pm \mathrm{i})u) = (A + tV \pm \mathrm{i})u, \tag{5}$$

$$S : \mathcal{G}(A + tV \pm \mathrm{i}) \to H, \quad S(u, (A + tV \pm \mathrm{i})u) = \zeta u, \quad |\zeta| \ll 1. \tag{6}$$

$\operatorname{Im} T$ closed, $\ker T = \{0\}$, $\|S\| \ll 1$. By Fredholm, $\operatorname{Im}(T + S)$ is closed, $\ker(T + S) = \{0\}$, and $\operatorname{codim}\operatorname{Im}(T + S) = -\operatorname{Ind}(T + S) = -\operatorname{Ind} T = \operatorname{codim}\operatorname{Im} T$. Thus $n_{\pm}(t)$ is constant in t.

□

Example 10. $H = H_0 + \gamma|x|^{-1}$ *on* $L^2(\mathbb{R}^3)$, $H_0 = -\triangle$, $\gamma \in \mathbb{R}$, $\mathcal{D}_{H_0} = \{u \in L^2 : \triangle u \in L^2\}$ *is s.a. with* $\mathcal{D}_H = \mathcal{D}_{H_0}$.

Proof. We prove the following estimate:

$$\int \frac{|\chi u|^2}{|x|^2} \lesssim \|\triangle u\|^2 + \|u\|^2, \tag{7}$$

then use Kato's theorem to conclude. □

Remark 5. *Remark of history: Kato also proved the many-body version of this theorem.*

Theorem 5.3 (Cayley transform)**.** *If A is symmetric, densely defined, then $T : (A + \mathrm{i})u \mapsto (A - \imath)u$ with $\mathcal{D}_T = (A + \mathrm{i})\mathcal{D}_A$ is isometric,* $\mathrm{Im}(I - T)$ *is dense,* $\ker(I - T) = \{0\}$ *and $\mathcal{D}_A = (I - T)\mathcal{D}_T$.*

Conversely, if T is isometric with $\mathrm{Im}(I - T)$ *dense and* $\ker(I - T) = \{0\}$*, then $A : (I - T)v \mapsto \mathrm{i}(I + T)v$ with $\mathcal{D}_A = (I - T)\mathcal{D}_T$ is symmetric and densely defined.*

Corollary 4. *Suppose A is symmetric and densely defined and closed, then A has a s.a. extension iff $n_+ = n_-$.*

Proof. By Cayley transfrom, A has a s.a. extension iff T has a unitary extension. This iff

$$\mathrm{codim}\mathcal{D}_T = \mathrm{codim}\,\mathrm{Im}\,T. \tag{8}$$

Since $\mathcal{D}_T = \mathrm{Im}(A + \mathrm{i})$, $\mathrm{Im}\,T = \mathrm{Im}\,\mathrm{i}(A + I) = \mathrm{Im}(A - \mathrm{i})$, we have the desired result. □

Theorem 5.4 (Spectral theorem I)**.** *There exists a unique operator-valued mapping $f \mapsto f(H) \in \mathcal{L}(\mathcal{H})$ for any s.a. operator H on a Hilbert space $\mathcal{H}$ and $f \in C(\mathbb{R})$ such that*

- $(fg)(H) = f(H)g(H)$*,*
- $f(H)^* = \overline{f}(H)$*,*
- *The functional calculus of $r_w(z) = (w - z)^{-1}$ is the resolvent, i.e. $r_w(H) = (H - wI)^{-1}$,*
- *If $\mathrm{supp} f$ is away from $\sigma(H)$, then $f(H) = 0$,*

Definition 5.5. *We say that $L \subset \mathcal{H}$ closed subspace is invariant under H, if $(H - z)^{-1}v \in L$ for any $v \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.*

Lemma 2. *$\mathcal{H}$ is separable, then $\exists L_j \subset H$ cyclic subspaces (i.e. $\exists u_j \in L_j$ such that $L_j = \overline{Span\{(H - z)^{-1}u_j : z \notin \mathbb{R}\}}$) such that $\mathcal{H} = \bigoplus_j L_j$ and $L_j \perp L_k$ for $j \neq k$.*

Theorem 5.6 (Spectral theorem II)**.** *H is s.a. on $\mathcal{H}$, $S = \sigma(H) \subset \mathbb{R}$, then there exists a μ finite measure on $S \times \mathbb{N}$, and a unitary operator $U : \mathcal{H} \to L^2(S \times \mathbb{N}, \mathrm{d}\mu)$ such that*

- *If $h : S \times \mathbb{N} \to \mathbb{R}$ is the function:*
$$h(\lambda, n) = \lambda, \tag{9}$$
(i.e. the multiplication by λ on copies indexed by n), then $\xi \in \mathcal{D}_H$ iff $hU(\xi) \in L^2(S \times \mathbb{N}, \mathrm{d}\mu)$, and for such ξ we have the diagonalization
$$UHU^{-1}\zeta = h\zeta, \tag{10}$$
for any $\zeta \in U(\mathcal{D}_H)$.
- *We have the functional calculus*
$$Uf(H)U^{-1}\zeta = f(h)\zeta. \tag{11}$$

In the case where we have a cyclic vector, i.e.

$$\mathcal{H} = \overline{\mathrm{Span}\,\{(z - H)^{-1}v : z \in \mathbb{C} \setminus \mathbb{R}\}} \tag{12}$$

Then the spectral theorem II can be simplified as follows:

Theorem 5.7 (Spectral theorem II′). *$\exists\mu$ a finite measure on S, $U : \mathcal{H} \to L^2(S, \mathrm{d}\mu)$ unitary such that*

- *$\xi \in \mathcal{D}_H$ iff $hU(\xi) \in L^2(S, \mathrm{d}\mu)$, h is the multiplication by λ on S, and*
- *we have the diagonalization*

$$UHU^{-1}\zeta = h\zeta, \tag{13}$$

  for any $\zeta \in U(\mathcal{D}_H)$.
- *we have the functional calculus.*