Convergence of Fourier Series

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1 Notations

Let f be a P-periodic function. We define the Fourier series of f as

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n x/P}, \text{ where } \hat{f}(n) = (f, e^{2\pi i n x}) = \frac{1}{P}\int_0^P f(x)e^{-2\pi i n x/P}\,\mathrm{d}x. \tag{1}$$

We also define the real Fourier series as

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right), \tag{2}$$

$$\text{where } a_n = \frac{2}{P}\int_0^P f(x)\cos\left(\frac{2\pi n x}{P}\right)\mathrm{d}x, \quad b_n = \frac{2}{P}\int_0^P f(x)\sin\left(\frac{2\pi n x}{P}\right)\mathrm{d}x. \tag{3}$$

The equivalence of the two definitions is given by

$$\hat{f}(n) = \begin{cases} a_0 & \text{if } n = 0, \\ \frac{a_n - ib_n}{2} & \text{if } n > 0, \\ \text{``}\frac{a_n - ib_n}{2}\text{''} = \frac{a_{-n} + ib_{-n}}{2} & \text{if } n < 0. \end{cases} \tag{4}$$

We define the partial sum of the Fourier series as

$$S_n = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi i k x/P} = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(\frac{2\pi k x}{P}\right) + b_k \sin\left(\frac{2\pi k x}{P}\right). \tag{5}$$

For simplicity, now we assume $P = 2\pi$, then the partial sum can also be written as

$$S_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-n}^{n} e^{\mathrm{i}k(x-t)}\right) f(t)\mathrm{d}t =: \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-t)f(t)\mathrm{d}t = \frac{1}{2\pi}[D_n * f](x). \tag{6}$$

Here, $D_n(x) = \sum_{k=-n}^{n} e^{\mathrm{i}kx}$ is the Dirichlet kernel, which can be expressed as

$$D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}. \tag{7}$$

Remark 1. *It is very easy to verify that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\mathrm{d}t = 1, \quad D_n(-t) = D_n(t), \quad D_n(t) = D_n(t+2\pi). \tag{8}$$

2 Riemann-Lebesgue Lemma

The Riemann-Lebesgue lemma states that

Theorem 2. *Let f be a P-periodic function in $L^1([0,P])$. Then*

$$\lim_{n\to\infty} \hat{f}(n) = 0. \tag{9}$$

More precisely, if $f \in L^1([0,P])$ (or we can understant it as $f \in \mathcal{R}([0,P])$ and is absolutely integrable), then

$$\lim_{\lambda\to\infty}\int_0^P f(x)\cos\lambda x\,\mathrm{d}x = 0, \quad \lim_{\lambda\to\infty}\int_0^P f(x)\sin\lambda x\,\mathrm{d}x = 0. \tag{10}$$

3 Fourier Localization Theorem

We will show that D_n is *localized around* $x = 0$ as $n \to \infty$. More precisely, we first rewrite the integral formula of S_n as

$$S_n = \frac{1}{2\pi}\int_0^{\pi}[f(x+t) + f(x-t)]D_n(t)\mathrm{d}t. \tag{11}$$

Here we extend f as a 2π-periodic function on $\mathbb{R}$. We note that for any $\delta > 0$,

$$\left|\frac{f(x+t)+f(x-t)}{\sin(t/2)}\right| \le \frac{|f(x+t)| + |f(x-t)|}{\sin(\delta/2)}, \quad t \in [\delta, \pi]. \tag{12}$$

Thus we can see that $\frac{f(x+t)+f(x-t)}{\sin(t/2)}$ is absolutely integrable on $t \in [\delta, \pi]$. By the Riemann-Lebesgue lemma, we have

$$\lim_{n\to\infty}\int_{\delta}^{\pi} D_n(t)[f(x+t)+f(x-t)]\mathrm{d}t = \lim_{n\to\infty}\int_{\delta}^{\pi}\sin\left((n+\frac{1}{2})x\right)\frac{f(x+t)+f(x-t)}{\sin(t/2)}\mathrm{d}t = 0. \tag{13}$$

In other words, we have

$$S_n = \frac{1}{2\pi}\int_0^{\delta}[f(x+t)+f(x-t)]D_n(t)\mathrm{d}t + o(1), \quad (n\to\infty), \tag{14}$$

We can use this to derive the pointwise convergence of the Fourier series.

4 Dini Convergence Theorem

We study the convergence of S_n at the point $x_0 \in \mathbb{R}$.

By the property of the Dirichlet kernel, we have

$$S_n - S = \frac{1}{2\pi}\int_0^\pi [f(x_0+t) + f(x_0-t) - 2S]D_n(t)\mathrm{d}t = \frac{1}{2\pi}\int_0^\delta [f(x_0+t) + f(x_0-t) - 2S]D_n(t)\mathrm{d}t + o(1). \quad (15)$$

To obtain $S_n \to S$, we want to show the integrability of the function $[f(x_0+t) + f(x_0-t) - 2S]/t$ on $[0,\delta]$. If this can be done, then we can apply the Riemann-Lebesgue lemma again to get the desired convergence. This is the essence of the Dini convergence theorem.

Theorem 3 (Dini Convergence Theorem)**.** *f is a 2π-periodic function in $L^1([-\pi,\pi])$ [1] and S_n is the n-th partial sum of the Fourier series of f.*

If there exists a $\delta > 0$ such that the function

$$\frac{f(x_0+t) + f(x_0-t) - 2S}{t} \quad (16)$$

is absolutely integrable on $[0,\delta]$, then we have

$$\lim_{n\to\infty} S_n(x_0) = S(x_0). \quad (17)$$

Proof. If the function $\frac{f(x_0+t)+f(x_0-t)-2S}{t}$ is absolutely integrable on $[0,\delta]$, then so is the function $\frac{f(x_0+t)+f(x_0-t)-2S}{\sin(t/2)}$. We can then apply the Riemann-Lebesgue lemma to get

$$\int_0^\delta [f(x_0+t) + f(x_0-t) - 2S]D_n(t) = \int_0^\delta \sin\left((n+\frac{1}{2})t\right)\frac{f(x_0+t) + f(x_0-t) - 2S}{\sin(t/2)}\mathrm{d}t \to 0, \quad (n\to\infty). \quad (18)$$

Together with eq. (15), we finish the proof. □

Once we have the Dini convergence theorem, we can easily show the pointwise convergence of the Fourier series.

Theorem 4. *If f is α-Lipschitz continuous at the point $x_0 \in \mathbb{R}$[2], then the Fourier series of S_n satisfies*

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0+0) + f(x_0-0)}{2}. \quad (19)$$

Proof. We verify the integrability condition of the Dini convergence theorem. We have

$$\frac{|f(x_0+t) + f(x_0-t) - f(x_0+0) - f(x_0-0)|}{t} \le \frac{2C}{t^{1-\alpha}}. \quad (20)$$

Note that $\frac{2C}{t^{1-\alpha}}$ is absolutely integrable on $[0,\delta]$. Thus the result follows from the Dini convergence theorem. □

Corollary 5 (Dirichlet's theorem)**.** *If f is a 2π-periodic function that is piecewise differentiable on $[-\pi,\pi]$, then for any $x_0 \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0+0) + f(x_0-0)}{2}. \quad (21)$$

[1] Again, this can be understood just as the Riemannian absolute integrability

[2] This means that there exists a constant $C > 0$ and a constant $\alpha > 0$ such that $|f(x) - f(y)| \le C|x-y|^\alpha$ for all x, y in a neighborhood of x_0. We also call this the α-Hölder condition.

5 Uniform Convergence of Fourier Series

If f is m-times differentiable on $[-\pi, \pi]$, with:

- **Periodic boundary condition:** $f^{(k)}(-\pi) = f^{(k)}(\pi)$ for all $0 \le k < m$ (continuous on the torus);
- **Continuity condition:** $f^{(m)}$ is piecewise continuous on $[-\pi, \pi]$.

Then we can obtain even stronger convergence on $[-\pi, \pi]$.

Theorem 6. *Let f be a 2π-periodic function that is m-times differentiable on $[-\pi, \pi]$, with* ***periodic boundary condition*** *and* ***continuity condition*** *as above. Then*

$$|f(x) - S_n(x)| \le \frac{\varepsilon_n}{n^{m-\frac{1}{2}}}, \tag{22}$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers that converges to 0 as $n \to \infty$ and does not depend on x. In particular, this implies that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$.

Proof.
- First, by the *periodic boundary condition*, we integrate by parts m times to obtain

$$\hat{f}(n) = \frac{1}{2\pi(n\mathrm{i})^m} \int_{-\pi}^{\pi} f^{(m)}(x) e^{-\mathrm{i}nx} \mathrm{d}x =: \frac{1}{(n\mathrm{i})^m} \gamma_n. \tag{23}$$

  Here

$$\gamma_n = \widehat{f^{(m)}}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(m)}(x) e^{-\mathrm{i}nx} \mathrm{d}x. \tag{24}$$

  By the Riemann-Lebesgue lemma and the *continuity condition*, we have $|\gamma_n| \to 0$ as $n \to \infty$.

- Second, by the Bessel inequality, we have

$$\sum_{n \in \mathbb{Z}} |\gamma_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f^{(m)}(x) \right|^2 \mathrm{d}x < \infty. \tag{25}$$

- Finally

$$\begin{aligned} |f(x) - S_n(x)| &= \left| \sum_{|k| \ge n+1} \hat{f}(k) \right| \overset{\text{eq. (23)}}{\le} \sum_{|k| \ge n+1} \frac{|\gamma_k|}{k^m} \\ &\overset{\text{Cauchy-Schwarz}}{\le} \left(\sum_{|k| \ge n+1} |\gamma_k|^2 \right)^{1/2} \left(\sum_{|k| \ge n+1} \frac{1}{k^{2m}} \right)^{1/2} \le \left(\sum_{|k| \ge n+1} |\gamma_k|^2 \right)^{1/2} \left(\int_n^{\infty} \frac{1}{x^{2m}} \right)^{1/2} \\ &=: \frac{\varepsilon_n}{n^{m-\frac{1}{2}}}. \end{aligned} \tag{26}$$

  Here $\varepsilon_n \to 0$ by the eq. (25).

□

6 Mean-Square Convergence of Fourier Series

The Dini convergence theorem tells us that the Fourier series *in general* does not pointwise converge for an integrable function f on $[-\pi, \pi]$. However, the L^2-Fourier theory will show that the Fourier series always converges in the mean-square sense for general integrable functions.

Proposition 7. *Note that the space $L^2([-\pi,\pi])$ of square-integrable functions on $[-\pi,\pi]$ is a Hilbert space with the inner product*

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\mathrm{d}x. \tag{27}$$

Using this inner product space structure, we have

$$S_n(f) = \mathcal{P}_{\mathcal{S}_n}[f], \tag{28}$$

where $\mathcal{P}_{\mathcal{S}_n}$ is the orthogonal projection onto the subspace $\mathcal{S}_n$ spanned by $\{e^{\mathrm{i}kx}\}_{k=-n}^{n}$. In particular, we have for any given complex numbers $A_{-n},\ldots,A_n$,

$$\|f - S_n(f)\|_{L^2} \le \left\| f - \sum_{k=-n}^{n} A_k e^{\mathrm{i}kx}\right\|_{L^2}, \tag{29}$$

In particular, if we take $A_k = 0$, we get the Bessel inequality:

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 \le \|f\|_{L^2}^2. \tag{30}$$

Here, we define the L^2-norm of the function f on $[-\pi,\pi]$ as

$$\|f\|_{L^2} = \sqrt{(f,f)} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{31}$$

Theorem 8 (Parseval's indentity)**.** *Let $f \in L^2([-\pi,\pi])$. Then the Fourier series gives us an isometry between $L^2([-\pi,\pi])$ and $\ell^2(\mathbb{Z})$, i.e.,*

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 = \|f\|_{L^2}^2. \tag{32}$$