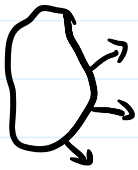
Partial Differential Equation

<u>Ref.</u> Lawrence C. Evans "Partial Differential Equation" 2nd Ed.

Review of Calculus and (linear) functional analysis

<u>Thrm.</u> (Gauss-Green Thm.) $\Omega \subset \mathbb{R}^n$ bdd. $\partial\Omega$ is C^1



Along $\partial\Omega$ define outward-pointing unit normal vect. field.

$\nu = (\nu^1, \dots, \nu^n)$

Let $u \in C^1(\bar{\Omega})$, we call $\frac{\partial u}{\partial \nu} := \nu \cdot \underline{Du}$

gradient vect.

$$Du = \left(\frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n}\right) = (u_{x^1}, \dots, u_{x^n})$$

- $\int_\Omega u_{x_i}\, dx = \int_{\partial\Omega} u \nu^i dS$
- (divergence theorem)

$$\int_\Omega \operatorname{div} \vec{u}\, dx = \int_{\partial\Omega} \vec{u} \cdot d\vec{S} := \int_{\partial\Omega} \vec{u} \cdot \nu\, dS$$

<u>Thrm.</u> (Integration by parts)

$u, v \in C^1(\bar{\Omega})$, then:

$$\int_\Omega u_{x_i} v\, dx = \int_{\partial\Omega} uv\, \nu^i dS - \int_\Omega v_{x_i} u\, dx$$

(follows from G-G thm. applied to uv)

<u>Thrm.</u> (Green's formulas) $u, v \in C^2(\bar{\Omega})$

- $\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$
- $\int_\Omega Du \cdot Dv\, dx = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \cdot u\, dS - \int_\Omega u \Delta v\, dx$
- $\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS$

Proof.

① By inte-by-parts, with u_{x_i} in place of u and $v=1$

we see:

$$\int_\Omega u_{x_i x_i}\, dx = \int_{\partial\Omega} u_{x_i} \nu^i\, dS$$

$$\Rightarrow \int_\Omega \Delta u\, dx = \sum_{i=1}^{n} \int_\Omega u_{x_i x_i}\, dx = \int_{\partial\Omega} \Big(\sum_{i=1}^{n} u_{x_i} \nu^i\Big) dS$$

$$= \int_{\partial\Omega} Du \cdot \nu\, dS = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS \qquad \square$$

② By inte-by-parts, with v_{x_i} in place of v:

$$\int_\Omega u_{x_i} v_{x_i}\, dx = \int_{\partial\Omega} u \cdot v_{x_i} \nu^i\, dS - \int_\Omega u\, v_{x_i x_i}\, dx$$

$$\Rightarrow \int_\Omega Du \cdot Dv\, dx = \sum_{i=1}^{n} \int_\Omega u_{x_i} v_{x_i}\, dx$$

$$= \int_{\partial\Omega} u \underbrace{\Big(\sum_{i=1}^{n} v_{x_i} \nu^i\Big)}_{= \frac{\partial v}{\partial \nu}} dS - \int u \Delta v\, dx \qquad \square$$

③ $$\int_\Omega Du \cdot Dv\, dx = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \cdot u\, dS - \int_\Omega u \Delta v\, dx \quad (\Theta)$$

$$\int_\Omega Du \cdot Dv\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v\, dS - \int_\Omega v \Delta u\, dx$$

Subtract, we get $\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial u} \Big(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\Big) dS$

$\square$

Thm. (Inverse function theorem)

$\Omega \subset \mathbb{R}^n$ Assume $f \in C^1(\Omega; \mathbb{R}^n)$ and $\det(Jf(x_0)) \neq 0$

Then $\exists$ an open set $U \subset \Omega$ with $x_0 \in U$ and an open set $V \subset \mathbb{R}^n$, with $f(x_0) \in V$, s.t.

- $f: U \to V$ is bijective
- $f^{-1}: V \to U$ is also C^1

$\}$ f is a C^1-differential homeomorphism

- If $f \in C^k$ then $f^{-1} \in C^k$

$f(x) = y \rightsquigarrow y = g(x)$

· If $f \in C^k$ then $f^{-1} \in C^k$

$f(x) = y \rightsquigarrow y = g(x)$
$F(x,y) = 0 \rightsquigarrow$ ~

Thrm. (Implicit functions theorem)

Assume $f \in C^1(\Omega; \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^{n+m}$

$(x, y) = (x^1, \dots, x^n, y^1, \dots, y^m)$

Denote $J_y f = \begin{pmatrix} f^1_{y_1} & \cdots & f^1_{y_m} \\ \vdots & \ddots & \vdots \\ f^m_{y_1} & \cdots & f^m_{y_m} \end{pmatrix}$, $\det J_y f(x_0, y_0) \neq 0$

Then $\exists$ an open set $U \subset \Omega$, with $(x_0, y_0) \in U$, an open set $V \subset \mathbb{R}^n$, with $x_0 \in V$, and a C^1 mapping $g: V \to \mathbb{R}^m$ s.t.

- $g(x_0) = y_0$
- $f(x, g(x)) = z_0$. If $(x, y) \in V$, $f(x,y) = z_0$ then $y = g(x)$
- If $f \in C^k$, then $g \in C^k$ $(k = 2, 3, \dots)$

Review of linear functional analysis

1. Normed vect. space and inner product space

Def. (Norm) $X/_{\mathbb{R}}$ a vect. space.
A mapping $\|\cdot\| : X \to [0, \infty)$ is called a norm, if

① $\|\lambda u\| = |\lambda| \cdot \|u\|$ (for all $\lambda \in \mathbb{R}$, $u \in X$)
② $\|u\| \geq 0$ and "=" holds iff $u = 0$
③ $\|u+v\| \leq \|u\| + \|v\|$ $(\forall u, v \in X)$

$(X, \|\cdot\|)$ is called a normed vect. space.

Def. (Banach Space)

$d(x,y) := \|x-y\|$
... "norm-topology" = "Strong-topology"

$(X, \|\cdot\|)$ is complete if

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"any <u>Cauchy sequence</u> in X converges"

$(\forall \varepsilon > 0 \ \exists N > 0 \text{ s.t. } \|u_k - u_l\| < \varepsilon \text{ for all } k, l \geq N)$

A <u>Banach space</u> X is a complete normed vect. space.

<u>Example</u>

① L^p space

$\Omega \subset \mathbb{R}^d$ is an open set, $1 \leq p \leq \infty$

$f : \Omega \to \mathbb{R}$ is a measurable functional. define:

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f|^p \, \underset{\substack{\uparrow \\ \text{L-measure on } \mathbb{R}^d}}{dx} \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_\Omega |f| & \text{if } p = \infty \end{cases}$$

$$L^p(\Omega) := \left\{ \text{the space of } f : \Omega \to \mathbb{R} \text{ measurable with } \underset{\substack{\uparrow \\ \text{"}L^p\text{- integrable"}}}{\|f\|_{L^p(\Omega)} < \infty} \right\}$$

<u>Prop.</u> • $L^p(\Omega)$ is Banach
• $L^p(\Omega)$ is <u>separable</u> if $1 \leq p < \infty$

② Hölder spaces

③ Sobolev spaces

<u>Def</u> (Hilbert spaces)

• (inner product space) Let X be a (real) vect. space
A mapping $(\cdot, \cdot) : X \times X \to \mathbb{R}$ is called an inner product, if:

① $(u, v) = (v, u)$

② $u \mapsto (u, v)$ is linear for fixed $v \in X$

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② $u \mapsto (u,v)$ is linear for fixed $v \in X$
③ $(u,u) \geq 0$ and $(u,u)=0$ iff $u=0$

$(X, (\cdot,\cdot))$ is called an inner product space

- The associated norm w.r.t. an inner product space: $\|u\| := \sqrt{(u,u)}$
- (Cauchy-Schwarz ineq.) $|(u,v)| \leq \|u\| \cdot \|v\|$
- (Hilbert space) A ~~Hilbert~~ space is a complete inner product space

Example.

- $\mathbb{R}^n$ $(x,y) := x^T y = \sum_{i=1}^n x^i y^i \rightsquigarrow \|\cdot\|_2$
$\sqrt{(x,x)} = \sqrt{\sum_{i=1}^n (x^i)^2}$
- $L^2(\Omega)$
$(f,g) := \int_\Omega fg \, dx \rightsquigarrow \|\cdot\|_{L^2(\Omega)}$
- The Sobolev space $H^1(\Omega)$

Def. (Orthogonality) $(X, (\cdot,\cdot))$

- $u, v \in X$ is orth. if $(u,v)=0$
- (Orthogonal system). (orthonormal) $\{w_k\}_{k \in I} \subset X$,
is called o.n. if $(w_k, w_l) = \delta_{kl}$
- (maximal o.n. system) $\{w_k\}_{k \in I} \subset X$
is o.n. system, with:
$(u, w_k) = 0 \ \forall k \in I \Rightarrow u=0$

If additionally I is countible (denoted as $\{w_k\}_{k=1}^\infty$)

Then we can write u as "Fourier Expansion":

as $\{w_k\}_{k=1}^{\infty}$

Then we can write u as "Fourier Expansion":

$$u = \underbrace{\sum_{i=1}^{\infty} (u, w_i)}\, w_i$$

(An separable Hilbert space must have a countible maximal o.n. system)

Parseval eq. states:

$$\|u\|^2 = \sum_{i=1}^{\infty} |(u, w_i)|^2$$

($L^2[0, 2\pi]$ $\underbrace{w_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}}$ is a maximal o.n. system)

2. Linear Operators on Banach spaces

<u>Def</u>. (Linear Operator) X, Y (real) Banach spaces

- (Linear Operators) A mapping $A: X \to Y$ is called <u>linear</u> provided $A(\lambda x + \mu y) = \lambda A(x) + \mu A(y)$ for all $\lambda, \mu \in \mathbb{R}$, $x, y \in X$
- $\ker A = \{u \in X : Au = 0\}$
  $\operatorname{Im} A = A(X) = \{v \in Y : v = Au \text{ for some } u \in X\}$
- (Bdd. linear operators) $L(X;Y)$ $L(X) := L(X;X)$
  $A: X \to Y$ linear oprator, A is bdd., if
  $$\|A\| := \sup_{0 \neq u \in X} \frac{\|Au\|}{\|u\|} = \sup_{\|u\|=1} \|Au\| = \sup_{\|u\| \le 1} \|Au\| < \infty$$

<u>Rmk</u>. It's easy to verify ~~that~~ a bdd. linear operator $A: X \to Y$ is conti.

- A bdd. linear operator $u^*: X \to \mathbb{R}$ is called a bdd. linear functional on X.

We write $X^* := \underline{L(X;\mathbb{R})}$ (the dual space of X)

Rmk. $\mathbb{R}^n$ $\quad (\mathbb{R}^n)^* = L(\mathbb{R}^n;\mathbb{R}) = M_{1\times n}(\mathbb{R})$

$\mathbb{R}^3 \ni (x,y,z)$ $\ell(\vec{x}) = 3x + 100y + 4z$

$$= \underbrace{(3 \quad 100 \quad 4)}_{\text{"}\ell\text{"} \in (\mathbb{R}^3)^*} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thrm (Riesz Representation theorem)

- (Notation) The "pairing" of X^* and X $\langle \cdot , \cdot \rangle$ is given by

$$\langle \underset{X^*}{\underset{\cap}{u^*}}, \underset{X}{\underset{\cap}{u}} \rangle := u^*(u)$$

$$\left(\text{Eg. } \langle \underset{(\mathbb{R}^3)^*}{\underset{\cap}{\ell}}, \underset{\mathbb{R}}{\underset{\cap}{\vec{x}}} \rangle = \ell(\vec{x}) = (3 \quad 100 \quad 4)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)$$

- (Riesz Representation theorem) "$X^* \simeq X$"

X Hilbert space, $\forall\, u^* \in X^*$, $\exists!\, u \in X$ s.t. $\underline{u^*(v)} = (v, u)$ for any $v \in X$

$$\left(\langle u^*, v \rangle = (v, u) \right)$$

$\langle v, u^* \rangle$

Eg. $\mathbb{R}^3$. ℓ as above. $\quad \ell(\vec{x}) = (\vec{x}, \vec{y}) = \vec{x}^T \vec{y}$

where $\vec{y} = \begin{pmatrix} 3 \\ 100 \\ 4 \end{pmatrix}$ $\quad = \vec{y}^T \vec{x}$

and $\|u^*\| = \|u\|$

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norm of operator $\|u^*\| := \sup_{0 \neq w \in X} \frac{|u^*(w)|}{\|w\|}$

Thrm/Def. (Adjoint Operator)

X Hilbert, $A \in L(X)$, $\exists!\ A^* \in L(X)$, s.t.

$(Au, v) = (u, A^* v)$ for all $u, v \in X$

A^* is called the adjoint operator of A.

Proof. (Exer) (Hint: use Riesz's Representation thrm.)

We say A is <u>self-adjoint</u> (symmetric) if $A^* = A$

Rmk. • $\|A^*\| = \|A\|$

• $(A^*)^* = A$

• $(AB)^* = B^* A^*$

Example. $X = \mathbb{R}^n$ $(x, y) = x^T y$

$\underline{A^* = A^T}$ (transpose of A)

$[(Ax, y) = (Ax)^T y = x^T(A^T y) = (x, A^T y)]$

Thrm. (Banach-Alaoglu) $(X, \|\cdot\|)$ Banach

• **Def.** (Weak convergence)

$\{u_k\}_{k=1}^\infty \subset X$ converges weakly to $u \in X$,

written $u_k \rightharpoonup u$ $(k \to \infty)$

if $\forall u^* \in X^*$, $u^*(u_k) \longrightarrow u^*(u)$ $(k \to \infty)$

if $\forall u^* \in X^*$, $\underset{\mathbb{R}}{\underset{\cap}{u^*(u_k)}} \longrightarrow u^*(u)$ $(k \to \infty)$

$\left(|u^*(u_k) - u^*(u)| = |u^*(u_k - u)| \leq \|u^*\| \cdot \underset{\downarrow\, 0\, (k\to\infty)}{\|u_k - u\|} \right)$

- Def (Reflexive Space) 自反

FACT A canonical isometric embedding: 等距

$J: X \longrightarrow X^{**}$

$X^{**} = (X^*)^* := L(X^*, \mathbb{R})$

$x \longmapsto J(x) \in X^{**}$

$J(x)(x^*) := x^*(x) = \underline{\langle x^*, x \rangle} = \langle x, x^* \rangle$ ($J(x)$)

$\|x\| = \|J(x)\|$

$X \cong J(X) \subseteq X^{**}$

We say X is reflexive if $J(X) = X^{**}$

$\left(X \overset{J}{\cong} X^{**} \right)$

Rmk. • A reflexive normed vect. space is Banach.

• Hilbert space must be reflexive.

- Thrm (Banach-Alaoglu)

X reflexive, $\{u_k\}_{k=0}^{\infty} \subseteq X$ a bdd. sequence

Then $\exists$ a subsequence $\{u_{k_\ell}\}_{\ell=0}^{\infty}$ of $\{u_k\}_{k=0}^{\infty}$ and some $u \in X$ s.t. $u_{k_\ell} \rightharpoonup u$

$\{u_k\}_{k=1}^{\infty}$ and some $u \in X$, s.t. $u_{k_l} \rightharpoonup u$

(closed unit ball of X must be <u>w-compact</u>)

<u>Rmk.</u> In particular, "a bdd. sequence in Hilbert space must have a weakly convergent subsequence"

<u>Example</u>. f.d.

- $X = L^p(\Omega)$ $(1 \le p < \infty)$, then

  $X^* = L^q(\Omega)$ $(1 < q \le \infty)$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

  $\Big(\forall\, l \in X^*, \exists !\, g \in L^q(\Omega)$, s.t.

  $$l(\underset{L^p}{f}) = \underline{\int_\Omega f g \, dx}$$

  Hölder's eq. $\underline{|l(f)| \le \underbrace{\left(\int_\Omega |g|^q dx\right)^{1/q}}_{\|g\|} \left(\int_\Omega |f|^p dx\right)^{1/p}}$

  $\|l\| = \|g\|_{L^q}$ $\Big)$

In particular:

① $L^p(\Omega)$ is reflexive if $1 < p < \infty$

$$L^p \xrightarrow{\text{dual}} L^q \xrightarrow{\text{dual}} L^p$$

$$L^1 \xrightarrow{\text{dual}} \underline{\underline{L^\infty}} \overset{\text{dual}}{\not\longrightarrow} L^1$$

② Any bdd. sequence in $L^p(\Omega)$ $(1<p<\infty)$ must have a weakly convergent subsequence

$\Rightarrow$ $\forall \{f_k\}_{k=1}^{\infty} \subset L^p(\Omega)$ with $\|f_k\|_{L^p} \le M$ $(\forall k)$

i.e. $\forall \{f_k\}_{k=0}^{\infty} \subseteq L^p(\Omega)$ with $\|f_k\|_{L^p(\Omega)} \le M$ ($\forall k$), M indep. of k

$\exists$ subsequence $\{f_{k_\ell}\}_{\ell=0}^{\infty}$ ~~s.t. $(f_{k_\ell} \rightharpoonup f)$~~ and some $f \in L^p$

$$\underline{\int_\Omega f_{k_\ell} g \, dx \to \int_\Omega f g \, dx} \quad (\ell \to \infty) \quad (\forall g \in L^q(\Omega))$$

($f_k \to f$ a.e. $\Longrightarrow$ $f_k \rightharpoonup f$

[$\left|\int_\Omega f_k g - \int_\Omega f g\right|$

$$\le \underbrace{\|f_k - f\|_{L^p(\Omega)}}_{\le 2M} \|g\|_{L^q(\Omega)}$$)

By DCT, $\lim_{k\to\infty} \left|\int_\Omega (f_k g - f g)\right|$

$$\le \int_\Omega \lim_{k\to\infty} \underbrace{|f_k g - f g|}_{\to 0 \text{ a.e. } (k\to\infty)}$$

$$= 0$$

Next time: Compact operators and their spectrum theory

(Some "great theorem" in linear functional analysis)