Some "great theorems" in linear functional analysis

Baire theorem, Banach-Steinhaus theorem, Open mapping theorem, Closed graph theorem.
[Hahn-Banach Theorem, Banach closed-Image theorem]

<u>Thm.</u> (Baire theorem) (X, d) complete metric space.
(e.g. Banach space)

- (Open Ver.) $\{O_n\}_{n\geq 1}$ a sequence of dense open subsets of X. Then $\bigcap_{n=1}^{\infty} O_n$ is also dense in X. ($\overline{O_n} = X$)
$$\left(\overline{\bigcap_{n=1}^{\infty} O_n} = X\right)$$
- (Closed Ver.) $\{F_n\}_{n\geq 1}$ a sequence of closed subsets with $\operatorname{int} F_n = \emptyset$, then $\operatorname{int} \bigcup_{n=1}^{\infty} F_n = \emptyset$.

<u>Corollary.</u> X metric space. $\{F_n\}_{n\geq 1}$ a sequence of closed subsets with $X = \bigcup_{n=1}^{\infty} F_n$
(in particular $\operatorname{int} \bigcup_{n=1}^{\infty} F_n \neq \emptyset$)

- If $\operatorname{int} F_n = \emptyset$ for $n \geq 1$, then X is NOT complete.
- If X is complete, then $\exists\, n_0$ s.t. $\operatorname{int} F_{n_0} \neq \emptyset$.

<u>Thm.</u> (Banach-Steinhaus Thm.) Uniformly bounded ness Priciple
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$X \xrightarrow{A_\alpha} Y$ $\{A_\alpha\}_{\alpha \in I}$ a family of linear operators from X to Y. Satisfying:

(from X to Y), Satisfying:
Banach — normed vect. space

$\{A_\alpha x\}_{\alpha\in I}$ bdd. for any $x\in X$

$\Longrightarrow \sup\limits_{\alpha\in I}\|A_\alpha\| < \infty$

[A_α bdd. linear operator $A_\alpha(B)$ bdd,
$\{A_\alpha x\}_{\alpha\in I}$ bdd. $\Longrightarrow \bigcup\limits_{\alpha\in I} A_\alpha(B)$ bdd.]

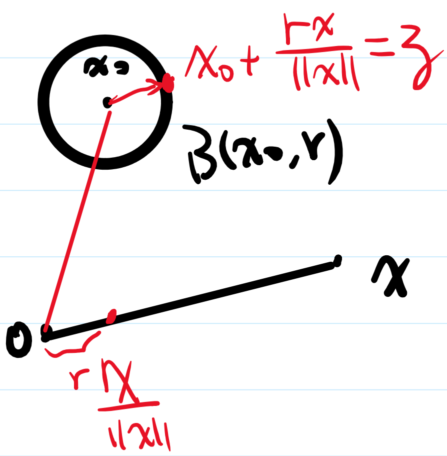
Proof. Define $F_n := \{x\in X : \sup\limits_{\alpha\in I}\|A_\alpha x\| \le n\}$

By assumption, $\bigcup\limits_{n=1}^{\infty} F_n = X$ ← Banach.

By the corollary of Baire's theorem, $\exists n_0$ s.t. $\operatorname{int}\overline{F_{n_0}} \neq \emptyset$

$\Longrightarrow \exists x_0 \in F_{n_0}$ and $r>0$ s.t. $B(x_0,r) \subset\subset F_{n_0}$

$(\overline{B(x_0,r)} \subset F_{n_0})$



Take $z = x_0 + \frac{rx}{\|x\|} \Longrightarrow x = \frac{\|x\|}{r}(z-x_0)$

$$\|A_\alpha x\| = \left\|A_\alpha\left(\frac{\|x\|}{r}(z-x_0)\right)\right\|$$

$$= \frac{\|x\|}{r}\|A_\alpha z - A_\alpha x_0\|$$

$z \in \overline{B(x_0,r)} \subset$, $x_0 \in F_{n_0}$

$$\le \frac{2n_0\|x\|}{\ }$$

$$\le \frac{2n_0\|x\|}{r} \quad \text{indep. of } \alpha \text{ and } x$$

$\cap$ F_{n_0}

$$\Rightarrow \sup_{\alpha\in I}\sup_{x\neq 0}\frac{\|A_\alpha x\|}{\|x\|} \le \frac{2n_0}{r} < \infty \Rightarrow \sup_{\alpha\in I}\|A_\alpha\| < \infty \quad \square$$

$\|A_\alpha\|$

Thm. (Open mapping thm.)

X, Y Banach $A \in L(X;Y)$

Assume A is surjective, ($\operatorname{Im} A = Y$)

Then A is an open-mapping. (maps any open set in X to an open set in Y)

Corollary. $X \xrightarrow{A} Y$, X, Y Banach. $A \in L(X;Y)$. A is an bijective.

Then $A^{-1} \in L(Y;X)$

Proof. Open mapping thm. $\Rightarrow$ A is an open mapping

$$\Rightarrow \boxed{\forall\, U \text{ open in } X.\ \underset{\substack{\| \\ (A^{-1})^{-1}(U)}}{A(U)} \text{ is open in } Y.}$$

$\Rightarrow A^{-1}$ conti. $\square$

E.g. (Application in PDE)

$\Omega \subset \mathbb{R}^d$ a bdd. open set

Consider a linear PDO:

$$D(x) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\boxed{\frac{\partial^\alpha}{\partial x^\alpha}}$$

$\geq m$

$D(x) := \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = \sum_{k=1}^n |\alpha_k| \le m}} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$

multi-index

$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

W.L.O.G. we may assume $a_\alpha(x) \in C^0(\overline{\Omega})$

Then $D: C^m(\overline{\Omega}) \longrightarrow C^0(\overline{\Omega})$

$$f \longmapsto Df(x) := \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le m}} a_\alpha(x) \frac{\partial^\alpha f}{\partial x^\alpha}(x) \in C^0(\overline{\Omega})$$

FACT. $(C^m(\overline{\Omega}), \|\cdot\|_{C^m})$ is Banach

$$\|f\|_{C^m} := \max_{|\alpha| \le m} \left\{ \sup_{x \in \overline{\Omega}} \left| \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| \right\}$$

$$\|f\|_{C^0} = \sup_{x \in \overline{\Omega}} |f(x)|$$

Consider a BVP: 边值问题

$$\begin{cases} Du = g & \text{in } \Omega \quad (\text{given } g \in C^0(\overline{\Omega})) \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

Assume $\exists!$ solution $u \in C^m(\overline{\Omega})$, then:

$$\|u\|_{C^m(\overline{\Omega})} \le C \|g\|_{C^0(\overline{\Omega})} \text{ for some } C > 0.$$

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indep. of g

Observe. $\|u\|_{C^m(\overline{\Omega})} \le C \|g\|_{C^0(\overline{\Omega})}$

$$\Longleftrightarrow \|u\|_{C^m(\overline{\Omega})} \le C \|Du\|_{C^0(\overline{\Omega})}$$

$(X := \{ f \in C^m(\overline{\Omega}) : f|_{\partial\Omega} = 0 \}, \|\cdot\|_{C^m(\overline{\Omega})})$

$\left(X := \{ f \in C^m(\bar{\Omega}) : f|_{\partial\Omega} = 0 \}, \ \|\cdot\|_{C^m(\bar{\Omega})}\right)$

$\subset \left(C^m(\bar{\Omega}), \ \|\cdot\|_{C^m(\bar{\Omega})} \right)$

FACT. Any closed subspace of a Banach space is complete.

$\left(Y := C^0(\bar{\Omega}), \ \|\cdot\|_{C^0(\bar{\Omega})}\right)$

$A : X \to Y, \quad f \mapsto Df$

- $A \in L(X;Y)$

$$\Big[\|Af\|_{C^0} = \Big\| \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le m}} a_\alpha(x) \frac{\partial^\alpha f}{\partial x^\alpha}(x) \Big\|_{C^0}$$

$$\le \max_{|\alpha| \le m} \|a_\alpha\|_{C^0} \left(\sup_{x \in \bar{\Omega}} \left| \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| \right)$$

$$\le \underbrace{M}_{= \max_{|\alpha|\le m} \|a_\alpha\|} \cdot \underbrace{\max_{|\alpha| \le m} \left(* \right)}_{= \|f\|_{C^m}} \Big]$$

$M \|f\|_{C^m}$

- A is bijective. [Follow from the assumption that "$\forall g \in C^0(\bar{\Omega}), \exists ! u \in X$ s.t. $Au = g$"]

By the corollary of OMT:

A^{-1} is conti. $\longrightarrow$ indep. of g

$\Rightarrow \|A^{-1} g\|_{C^m} \le C \|g\|_{C^0}$

Take $g = Au$, then we have

$\|u\|_{C^m} \le C \cdot \|g\|_{C^0} = C \|Du\|_{C^0}$ □

Thm. (Closed graph thm.)

X, Y Banach spaces $A: X \to Y$ a linear operator.
Then A is conti. iff A has a closed graph

$$\left(\begin{cases} Ax_n \to y \\ x_n \to x \end{cases} \Longrightarrow Ax = y \right)$$

$$Gr(A) = \{ (x, Ax) : x \in X \} \subset X \times Y.$$

Compact operators and their spectrum theory

Def. (Compact operator)

X, Y normed vect. spaces

We say $T: X \to Y$ is compact if

$\forall\, U \subset X$ bdd., $\overline{T(U)}$ is compact in Y.

Prop. • T compact $\Rightarrow T \in L(X; Y)$

• X Hilbert space, then

T compact $\Longleftrightarrow T^*$ compact

• If $\underline{\dim \operatorname{Im} T < \infty}$, then T is compact.

• Denote $K(X) := \{$ compact operators $X \to X \}$
then $K(X)$ is a closed ideal of $\underline{\underline{L(X)}}$.

then $K(X)$ is a $\underbrace{\text{closed ideal}}_{\text{(two-sided)}}$ of $L(X)$.

- X, Y Banach, $T: X \to Y$ compact.

  If $\operatorname{Im} T$ is closed, then $\dim \operatorname{Im} T < \infty$.

- X Banach, $T \in L(X)$ compact,

  then $\dim \ker(\lambda \mathrm{Id} - T) < \infty$ for $\lambda \neq 0$.

- $\dim X = \infty$ Banach. $T \in L(X)$ compact

  $\Rightarrow 0 \in \underbrace{\sigma(T)}_{\text{spectrum of } T} := \{\lambda : \lambda \mathrm{Id} - T \text{ is not invertible}\}$

Def (Self-adjoint compact operator)

X is Hilbert. $T \in L(X)$ compact with $T = T^*$.

Def/Prop. X Hilbert (over $\mathbb{R}$ or $\mathbb{C}$) $\mathbb{K}$, T is a self-adj. compact operator

- (Eigenvalue) $\exists\, 0 \neq p \in X$ and $\lambda \in \mathbb{C}$, s.t. $Tp = \lambda p$, then:

  p is called an eigenvector of T

  λ is called an eigenvalue of T

- Any eigenvalue of T is real.

  [Proof. $Tp = \lambda p \Rightarrow (Tp, p) = (\lambda p, p) = \lambda \|p\|^2$]

  $(Tp, p) = (p, Tp) = (p, \lambda p) = \bar{\lambda} \|p\|^2$

$(p, Tp) = (p, \lambda p) = \bar{\lambda} \|p\|^2$
$\Downarrow$
$\lambda = \bar{\lambda}$

- $\lambda_1 \neq \lambda_2$ are two eigenvalues of T, with eigenvectors p_1, p_2 resp., then
$$(p_1, p_2) = 0.$$
[Proof. Exer.]

- $$\|T\| = \sup_{x \neq 0} \frac{|(Tx, x)|}{\|x\|^2} = \sup_{\|x\|=1} |(Tx, x)|$$
Proof. Exer. (Hint: use $\displaystyle \|T\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|(Tx, y)|}{\|x\| \cdot \|y\|}$))

Thm. X Hilbert. T is a compact self-adjoint operator, then $\exists$ an eigenvalue λ of T s.t.
$$|\lambda| = \|T\|.$$
Proof. Take $\|x_n\| = 1$ s.t. $|(Tx_n, x_n)| \to \|T\|$

$\left\{\begin{array}{l} T \text{ compact} \\ \{(Tx_n, x_n)\} \text{ bdd.} \end{array}\right. \Rightarrow \exists$ a subsequence $\{x_n\}$ (w.l.o.g.) s.t.

$\left\{\begin{array}{l} Tx_n \to y \quad \text{for some } y \\ (Tx_n, x_n) \to \mu \quad \text{for some } \mu. \end{array}\right.$

Write $x_m = -\frac{1}{\mu}(Tx_m - \mu x_m) + \frac{1}{\mu} Tx_m$

CLAIM $\|Tx_m - \mu x_m\| \to 0$

CLAIM. $\|Tx_m - \mu x_m\| \to 0$

$$0 \le \|Tx_m - \mu x_m\|^2 = \|Tx_m\|^2 + \|\mu x_m\|^2 - 2\mu(Tx_m, x_m)$$

$$\le \|T\|^2 + \mu^2 - 2\mu(Tx_m, x_m)$$

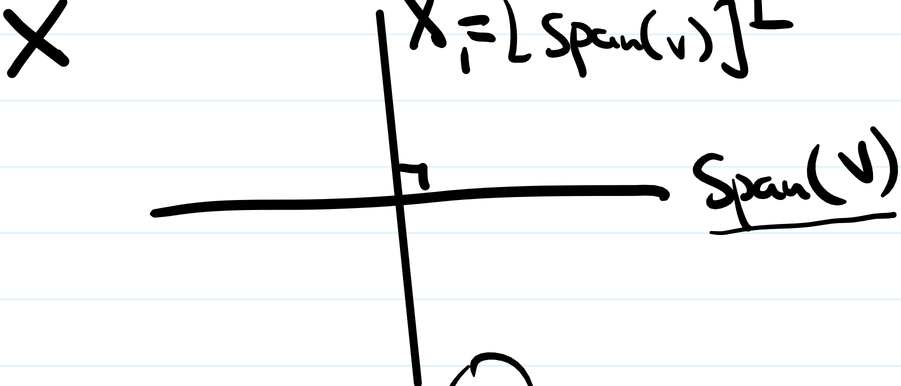
$$\le 2\|T\|^2 - 2\mu\underbrace{(Tx_m, x_m)}_{\to\mu} \to 2\|T\|^2 - 2\mu^2 = 0$$

$$\Rightarrow x_m \to \frac{1}{\mu} y \quad (m \to \infty)$$

$$\Rightarrow Ty = \lim_{m\to\infty} \mu T x_m = \mu \lim_{m\to\infty} T x_m = \mu y$$

with $|\mu| = \|T\|$. $\square$

Rmk. We can similarly find other eigenvectors



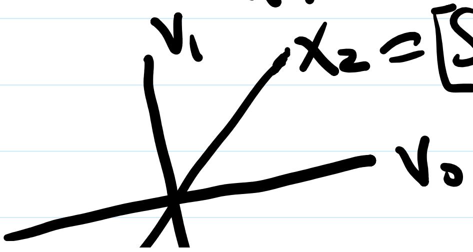
$T|_{X_1} : X_1 \to X_1$

$\Big[\; X_1 = \{x \in X : (x, v) = 0\} \qquad x \in X_1$

$\quad (\underbrace{Tx}_{\in X_1}, v) = (x, Tv) = (x, \lambda v) = \lambda(x, v) = 0 \;\Big]$

FACT. $T_1 := T|_{X_1}$ is a compact self-adj. operator on X_1 $\rightsquigarrow$ λ_1 s.t. $|\lambda_1| = \|T_1\|$

$X_2 = [\mathrm{Span}(v_0, v_1)]^\perp$



$\lambda_2, \lambda_3, \lambda_4, \dots$

(Spectrum theorem for compact self-adj. operators)

Thm. X Hilbert $T \in L(X)$ a self-adj. compact operator. Assume $\dim \operatorname{Im} T = N \in \mathbb{N} \cup \{\infty\}$

then:

- $\exists \{\lambda_n\}_{n=1}^N$ of T and eigenvectors $\{p_n\}_{n=1}^N$,

  s.t. $|\lambda_1| = \|T\| = \sup\limits_{\|x\|=1} |(Tx, x)|$

  $$|\lambda_2| = \sup_{\substack{\|x\|=1 \\ x \in \operatorname{Span} p_1^\perp}} |(Tx, x)|$$

  $$|\lambda_3| = \sup_{\substack{\|x\|=1 \\ x \in \operatorname{Span}(p_1, p_2)^\perp}} |(Tx, x)|$$

  $\vdots$ where $\underline{\lambda_n \neq 0}$

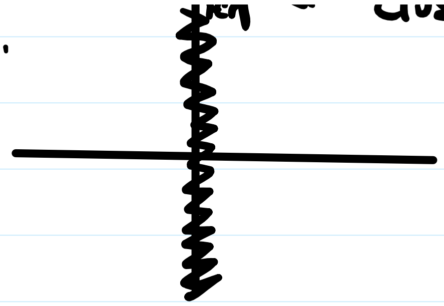
- If $N = \infty$, then $\lim\limits_{n \to \infty} \lambda_n = 0$
- $\forall x \in X$, $\underline{Tx = \sum\limits_{k=1}^N \lambda_k (x, p_k) p_k}$ ("diagonalization")
- Let $\lambda \neq 0$ be an eigenvalue of T, then we have $n \geq 1$, s.t. $\lambda_n = \lambda$, and

  $I(\lambda) = \{n \geq 1 : \lambda_n = \lambda\}$ is finite,

  $\underline{\operatorname{Span}\{p_n\}_{n \in I(\lambda)}}$ is finite dimensional.

Rmk. $\ker T \leftarrow$ closed $\quad X = \ker T \oplus \ker T^\perp$

Rmk. closed



$X = \ker T \oplus \ker T^{\perp}$

$\ker T^{\perp} = \operatorname{Im} T$ $\left[\underline{\text{FACT}}.\ \underbrace{\ker T^{\perp} = \operatorname{Im} T^{*}}_{= \operatorname{Im} T}\right]$

$$\ker T = \left(\operatorname{Span}(p_n)_{n=1}^{N}\right)^{\perp}$$

$$X = \underline{\underline{\ker T}} \oplus \underline{\left(\bigoplus_{n=1}^{N} \operatorname{Span}(p_n)\right)}$$

gives a diagonalisation of T. 写法 eigenspaces of non-zero eigenvalues

$$X \ni x := \underline{u} + \sum_{i=1}^{N} c_i p_i$$

$$Tx = \sum_{i=1}^{N} \lambda_i \underline{\underline{c_i}}\, p_i = \sum_{i=1}^{N} \lambda_i (x_i, p_i) p_i$$

[Ref. C.G. Ciarlet. LNLFA]

Setting. X complex Banach space. $T \in L(X)$ compact.

$\underline{\text{Eg}}$. $X = \ell^2(\mathbb{N}) = \left\{(x_1, x_2, \cdots) : \sum_{k=1} |x_k|^2 < \infty\right\}$

Consider $R : X \to X$ $\quad (x_1, x_2, \cdots) \mapsto \underbrace{(0, x_1, x_2, \cdots)}_{\ell^2}$

R is a bdd. linear operator.

$\underline{\text{FACT}}$. $\nexists\ x \neq 0$ and λ s.t. $Rx = \lambda x$

$\underline{Rx = \lambda x} \Rightarrow (0, x_1, x_2, \cdots) = (\lambda x_1, \lambda x_2, \cdots)$

$Rx = \lambda x \Rightarrow (0, x_1, x_2, \cdots) = (\lambda x_1, \lambda x_2, \cdots)$

$\Rightarrow \lambda x_1 = 0$

① if $\lambda = 0 \Rightarrow x_n = 0 \ (n \geq 1)$

② if $\lambda \neq 0 \Rightarrow x_1 = 0 \qquad \lambda x_2 = x_1 = 0 \Rightarrow x_2 = 0$

$\lambda x_3 = x_2 \Rightarrow x_3 = 0 \Rightarrow \cdots \ x = 0$

Def (Spectrum) $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda Id - T$ is NOT invertible$\}$

$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda Id - T$ is NOT injective$\}$

point spectrum(点谱) (corresponding to "eigenvalue")

$= \{\lambda \in \mathbb{C} : Tx = \lambda x$ for some $x \neq 0\}$

$\sigma(T) = \sigma_p(T) \cup \{\lambda \in \mathbb{C} : \lambda Id - T$ is NOT surjective$\}$

rest spectrum(剩余谱)

$\rho(T) := \mathbb{C} \setminus \sigma(T)$ the resolvent set of T

$= \{\lambda \in \mathbb{C} : \lambda Id - T$ is invertible$\}$ (预解集)

Next time

Spectrum theory for compact operators on Banach spaces (Fredholm theory)

Notations in PDE.