Review

Thm. Let X be a Hilbert space, $T\in L(X)$ is a self-adjoint compact operator. Denote $\dim \operatorname{Im} T = N \in \mathbb{N}\cup\{\infty\}$

- $\exists$ eigenvalues $\{\lambda_n\}_{n=1}^{N}\subset\mathbb{R}$ of T and eigenvectors $\{p_n\}_{n=1}^{N}$ of T $\left((p_i, p_j)=\delta_{ij}\right)$.
- If $N=\infty$, then $\lambda_n\to 0\ (n\to\infty)$
- $X = \ker T \oplus \underbrace{\ker T^{\perp}}_{=\operatorname{Im} T} = \underline{\ker T \oplus \left(\bigoplus_{n=1}^{N} \operatorname{Span}(p_n)\right)}$,

$$\forall x\in X,\quad Tx = \sum_{k=1}^{N}\lambda_k (x,p_k)\,p_k$$

$$x = \underbrace{u}_{\in \ker T} + \sum_{k=1}^{N}(x,p_k)\,p_k$$

- Let $\lambda\neq 0$ be an eigenvalue of T, then $\exists\, n\geq 1$ s.t. $\lambda_n=\lambda$ and $I(\lambda)=\{n\geq 1 : \lambda_n=\lambda\}$ is finite (and $\ker(\lambda \mathrm{Id}-T)=\operatorname{Span}(p_n)_{n\in I(\lambda)}$ is f.d.)

Setting $X/\mathbb{C}$ is a Banach space, $T\in L(X)$ a compact operator.

$$\sigma(T) := \{\lambda\in\mathbb{C} : \lambda \mathrm{Id}-T \text{ is NOT invertible}\}$$

$$\sigma_p(T) := \{\lambda\in\mathbb{C} : \lambda \mathrm{Id}-T \text{ is NOT injective}\}$$

$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda Id - T$ is NOT invertible$\}$

$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda Id - T$ is NOT injective$\}$
$= \ker(\lambda Id - T)$

$\rho(T) := \mathbb{C} \setminus \sigma(T)$ the resolvent set of T.

Thm. • $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists and
$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_n \|T^n\|^{\frac{1}{n}}$

denote $r(T) := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ (the spectrum radius of T)

• ② $\sigma(T)$ is non-empty and compact in $\mathbb{C}$.

$[\underbrace{\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\}}_{\Leftarrow}]$

① $\rho(T)$ is open

Proof • [Hint $n = q(n) n_0 + r(n)$ $(0 \le r_n \le n_0 - 1)$

denote $a = \inf_n \|T_n\|^{1/n}$, $\forall \varepsilon > 0, \exists n_0$ s.t. $a + \varepsilon > \|T_{n_0}\|^{\frac{1}{n_0}}$

$$\|T^n\|^{\frac{1}{n}} = \|\underbrace{T^{q(n) n_0 + r(n)}}\|^{\frac{1}{n}}$$

$$\le \|T\|^{n_0 \cdot \frac{q(n)}{n} \cdot n_0 \cdot \frac{1}{n_0}} \|T\|^{\frac{r(n)}{n}} \to 1 \quad (\|AB\| \le \|A\| \cdot \|B\|)$$

$\frac{q(n)}{n} \cdot n_0 \to 1$

$$\Rightarrow \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \le \|T^{n_0}\|^{\frac{1}{n_0}} < a + \varepsilon = \inf_n \|T_n\|^{\frac{1}{n}} + \varepsilon$$

$$\Rightarrow \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \le \inf_n \|T_n\|^{\frac{1}{n}} \le \liminf_{n\to\infty} \|T_n\|^{\frac{1}{n}} \le \limsup_{n\to\infty} \|T_n\|^{\frac{1}{n}}]$$

• Take $\lambda_0 \in \rho(T)$, $h \in \mathbb{C}$

$$(\lambda_0 + h)Id - T = \underset{\substack{\downarrow \\ \text{invertible}}}{(\lambda_0 Id - T)} + \underset{\sim}{hId}$$

$$\rho(T) = \{\lambda \in \mathbb{C} : \underset{\sim}{\lambda Id - T} \text{ is invertible}\}$$

λ_0

FACT. $K : X \to X$ invertible

$\Rightarrow K - A$ is invertible if $\underline{\|K^{-1}A\| < 1}$

$$(K-A)^{-1} = (K(Id - K^{-1}A))^{-1} = \left(\sum_{i=0}^{\infty} (K^{-1}A)^i\right) K^{-1}$$

$$(1-x)\left(\underline{1-x}\right)^{-1} = 1 + \underset{\substack{\sim \\ <1}}{x} + x^2 + \cdots = 1.$$

$(1-x)$

Take $|h| < \dfrac{1}{\|\lambda_0 Id - T\| > 0}$. then $\lambda_0 + h \in \rho(T)$. $\square$

• **CLAIM**. $\sigma(T) \subseteq \overline{D(0, \|T\|)}$

Only need to show that if $\lambda \in \mathbb{C}$, $|\lambda| > \|T\|$, then $\lambda \in \rho(T) \longrightarrow \lambda Id - T$ is invertible

$$(\lambda Id - T) = \lambda \underset{\substack{\Downarrow \\ \text{invertible}}}{\left(Id - \lambda^{-1}T\right)} \quad \left(|\lambda| > \|T\| \Rightarrow \left\|\tfrac{T}{\lambda}\right\| < 1\right)$$

$\Rightarrow \lambda \in \rho(T)$ $\square$

$\Rightarrow \sigma(T)$ compact.

(In fact, $\sigma(T) \subseteq \overline{D(0, r(T))}$ (Exer.))

• Take a bdd. linear functional $\xi \in L(X)^*$

$$\varphi : \rho(T) \longrightarrow \mathbb{C}$$

$$\lambda \mapsto (\lambda Id - T)^{-1} \overset{\xi}{\longmapsto} \xi[(\lambda Id - T)^{-1}]$$

$$\lambda \longmapsto \underbrace{(\lambda \mathrm{Id}-T)^{-1}}_{\mathcal{L}(X)} \overset{\xi}{\longmapsto} \xi\left[(\lambda \mathrm{Id}-T)^{-1}\right]$$

$\rightsquigarrow$ operator-valued holomorphic function.

CLAIM. φ is holomorphic.

Take $\lambda_0 \in \rho(T)$, $|h| < \|(\lambda_0 \mathrm{Id}-T)^{-1}\|^{-1}$

$$((\lambda_0+h)\mathrm{Id}-T)^{-1}$$

$$= (\lambda_0 \mathrm{Id}-T + h\mathrm{Id})^{-1}$$

$$= \left[(\lambda_0 \mathrm{Id}-T)(\mathrm{Id} + h(\lambda_0 \mathrm{Id}-T)^{-1})\right]^{-1}$$

$$= (\mathrm{Id}+h(\lambda_0 \mathrm{Id}-T)^{-1})^{-1}(\lambda_0 \mathrm{Id}-T)^{-1}$$

$$= \left[\sum_{k=0}^{\infty}(-1)^k h^k \cdot (\lambda_0 \mathrm{Id}-T)^{-k}\right](\lambda_0 \mathrm{Id}-T)^{-1}$$

$$\Rightarrow \xi\left(((\lambda_0+h)\mathrm{Id}-T)^{-1}\right) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\xi\left[(\lambda_0 \mathrm{Id}-T)^{-k-1}\right]}_{\in\, \mathbb{C}} h^k$$

In other words

"If $|h| < \dfrac{1}{\|(\lambda_0 \mathrm{Id}-T)^{-1}\|}$, then

$$\varphi(\lambda_0+h) = \sum_{k=0}^{\infty}(*)\, h^k$$

$\Rightarrow \varphi$ is hol. on $\rho(T) \subset \mathbb{C}$.

In fact

$$\varphi(\lambda) = \xi\left((\lambda \mathrm{Id}-T)^{-1}\right) = \xi\left(\sum_{n\geq 0} T^n \lambda^{-n-1}\right)$$

$$= \sum_{n\geq 0} \frac{\xi(T^n)}{\lambda^{n+1}} \quad \|\xi\|.\|T\|^n \quad \|\xi\|\,\|T\|^n$$

$$= \sum_{n\geq 0} \frac{\quad}{\lambda^{n+1}}$$

$$\Rightarrow |\varphi(\lambda)| \leq \sum_{n\geq 0} \frac{\|\xi\|\cdot\|T\|^n}{|\lambda|^{n+1}} = \sum_{n\geq 0} \frac{\|\xi\|}{|\lambda|}\left(\frac{\|T\|}{|\lambda|}\right)^n$$

$$= \frac{\|\xi\|}{|\lambda|}\cdot\frac{1}{1-\frac{\|T\|}{|\lambda|}} \to 0 \quad (\lambda\to\infty)$$

$\Rightarrow$ If $\rho(T)=\mathbb{C}$, then by Liouville's theorem,

$\varphi(\lambda)\equiv 0$

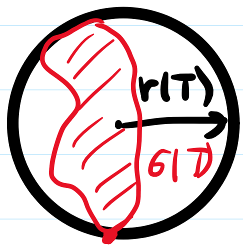
$$\Rightarrow \forall \xi \in (L(X))^*, \quad \xi\left((\lambda Id - T)^{-1}\right)=0$$

$$\langle \xi, (\lambda Id - T)^{-1}\rangle = 0 \quad \text{for any } \xi \in L(X)^*$$

$\Rightarrow (\lambda Id - T)^{-1} = 0$, a contradiction.

Therefore $\sigma(T)\neq\emptyset$. $\square$

Rmk. $r(T) = \max\limits_{\lambda\in\sigma(T)} |\lambda|$



Thm. $\sigma(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$

In other words, for $\lambda\neq 0$.

$\lambda Id - T$ surjective $\Leftrightarrow$ $\lambda Id - T$ injective

$Im(\lambda Id - T) = X \Leftrightarrow \ker(\lambda Id - T) = \{0\}$

Lem. 1 • Recall for $\lambda\neq 0$, $\dim\ker(\lambda Id - T)<\infty$

$\forall n\geq 1$ $\dim\ker(\lambda Id - T)^n < \infty$

• $Im(\lambda Id - T)^n$ is closed. (omit.)

Lem. 2 $\exists n_0$ s.t. $\ker(\lambda Id - T)^{n_0} = \ker(\lambda Id - T)^{n_0+1}$

Lem. 2 $\exists\, n_0$ s.t. $\ker(\lambda Id - T)^{n_0} = \ker(\lambda Id - T)^{n_0+1}$

$\Leftrightarrow \operatorname{Im}(\lambda Id - T)^{n_0} = \operatorname{Im}(\lambda Id - T)^{n_0+1}$

Proof. ①

- $K_n := \ker(\lambda Id - T)^n$

  K_n is closed. $\Rightarrow$ K_n is Banach

- $\operatorname{Im} T|_{K_n} \subseteq K_n$ $\left(x \in K_n \Rightarrow \underbrace{(\lambda Id - T)^n T x}_{= T(\lambda Id - T)^n x = 0}\right)$

  (K_n is T-invariant)

- $(\lambda Id - T)^n = \lambda^n Id - S$ $\left(S = \lambda^{n-1} T - \cdots (-1)^n T^n\right)$

  CLAIM. S is compact. $S = T\left(\underbrace{\lambda^{n-1} Id - \cdots (-1)^n T^{n-1}}_{\in L(X)}\right)$

  (Recall. $K(X)$ is a closed ideal of $L(X)$)

  $\lambda^n \neq 0$ $\Rightarrow \dim \underbrace{\ker(\lambda^n Id - S)}_{= K_n} < \infty$. $\square$

(Ref. Rudin "Functional Analysis")

② FACT. Consider $T^* : X^* \to X^*$

$T^* y^*(x) = y^*(Tx)$, $y^* \in X^*$

$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle$

$[(x, T^* y) = (Tx, y)]$

$(\operatorname{Ker}(\lambda Id - T^*)^n)^{\perp} = \operatorname{Im}(\lambda Id - T)^n$

$$\left(\operatorname{Ker}(\lambda Id - T^*)^n\right)^{\perp} = \overline{Im(\lambda Id - T)^n}$$

annihilator "零化子"

$$= Im(\lambda Id - T)^n$$

Therefore, we only need to prove that

$\exists\, n_0$ s.t. $K_{n_0} = K_{n_0+1}$

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots$$

Otherwise, $\forall n \geq 1$, $K_n \subsetneq K_{n+1}$

FACT. Take $x_n \in K_{n+1}$, $\|x_n\| = 1$, with $\operatorname{dist}(x_n, K_n) \geq \frac{1}{2}$

Consider $\{Tx_n\}$. By T compact, $\{Tx_n\}$ has a convergent subsequence.

$$\|Tx_m - Tx_n\| = \|Tx_m - \lambda x_m + \lambda x_m \underbrace{- Tx_n}_{K_n}\|$$

(Assume $m > n$, $K_n \subset K_{m-1} \subsetneq K_m$)

$$= |\lambda| \, \|\underset{\substack{\cap \\ K_m}}{x_m} + \lambda^{-1}(\underbrace{Tx_m - \lambda x_m}_{= -(\lambda Id - T)x_m \,\in\, K_{m-1}} \underset{\substack{\cap \\ K_{m-1}}}{- Tx_n})\|$$

$$\geq \frac{|\lambda|}{2}$$

$\operatorname{dist}(x_m, K_{m-1}) \geq \frac{1}{2}$

which contradicts to $\{Tx_n\}$ has a convergent subsequence. □

Take $\lambda \in \sigma(T) \setminus \{0\}$

convergent subsequence.

Proof of our main thm. Take $\lambda \in \sigma(T)\setminus\{0\}$, assume $\lambda \notin \sigma_p(T)\setminus\{0\}$

Lem2 $\Rightarrow \exists\, n_0$, s.t. $\operatorname{Im}(\lambda Id - T)^{n_0} = \operatorname{Im}(\lambda Id - T)^{n_0+1}$

$\forall x,\ \exists y$, s.t. $(\lambda Id - T)^{n_0} x = (\lambda Id - T)^{n_0+1} y$

$= (\lambda Id - T)^{n_0}(\lambda y - Ty)$

$\lambda \notin \sigma_p(T) \Rightarrow \lambda Id - T$ is injective,

$$\Rightarrow x = \lambda y - Ty = (\lambda Id - T)y$$

$\Rightarrow \lambda Id - T$ is surjective. $\Rightarrow \lambda \notin \sigma(T)$, a contradiction.

$$\sigma_p(T)\setminus\{0\} = \sigma(T)\setminus\{0\}.$$

In other words. If $\lambda \neq 0$, then $\lambda I - T$ is surj. $\Leftrightarrow$ is inj. $\square$

Notations Assume $\Omega \subset \mathbb{R}^n$, $x \in \Omega$.

Multi index Notation:

(a) $\alpha = (\alpha_1, \cdots, \alpha_n)$ where α_i is a nonnegative integer, is called a multiindex with order $|\alpha| = \sum_{i=1}^n \alpha_i$

(b) α is a multiindex,

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x)$$

$\alpha = (1,1)$. $$D^\alpha u(x) = \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = \partial_{x_1}\partial_{x_2} u(x) = u_{x_1 x_2}(x)$$

(c) If k is a non-negative integer, then:

$D^k u(x) := \{ D^\alpha u(x) \mid |\alpha| = k \}$ is the set of all k-order partial derivatives.

$$\# D^k u(x) = n^k$$

E.g. Gradient vect. $Du = (u_{x_1} \cdots u_{x_n})$

Hessian martix.

$$D^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}_{n \times n}$$

$$\Delta u = \mathrm{Tr}\, D^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

(d) vector-valued functions

- $\vec{u} : \Omega \to \mathbb{R}^m$, $\vec{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}$, we define

$$\vec{u} = \begin{pmatrix} \vdots \\ u^m \end{pmatrix}$$

$D^k\vec{u} = \{D^\alpha \vec{u} : |\alpha| = k\}$ as before.

- gradient matrix (Jacobian matrix)

$$D\vec{u} = \left(\frac{\partial \vec{u}}{\partial x_1} \cdots \frac{\partial \vec{u}}{\partial x_n}\right) = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \cdots & \frac{\partial u^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^m}{\partial x_1} & \cdots & \frac{\partial u^m}{\partial x_n} \end{pmatrix}_{m\times n}$$

- If $m=n$, we have:

$$\operatorname{div}\vec{u} = \operatorname{Tr} D\vec{u} = \sum_{i=1}^n u^i_{x_i} = \text{"divergence of } \vec{u}\text{"}$$

divergence

$$\int_\Omega \operatorname{div}\vec{u}\, dx = \int_{\partial\Omega} \vec{u}\cdot\nu\, dS$$
$$(\vec{u}\cdot d\vec{S})$$

Def. (Partial Differential Equation)

Fix an integer $k \geq 1$ and let $\Omega \subset \mathbb{R}^n$ open.

An expression of the form:

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \quad (*)$$

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0 \quad (*)$$

is called a k-th order PDE, where

$$F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given, and $u: \Omega \to \mathbb{R}$ is unknown.

called a solution of $(*)$

Rmk. We solve the PDE if we find all u satisfying $(*)$ (possibly among those satisfying certain boundary conditions on $\partial\Omega$). In practice, we may not be able to solve some PDEs, instead, we only deduce the existence or other properties of solutions.

Def.

- The PDE $(*)$ is called linear if it has the form:

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

The linear PDE is homogeneous if $f(x) \equiv 0$

- The PDE $(*)$ is semilinear if it has the form:

the form:

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u + a_0 (D^{k-1}u, \dots, Du, u, x) = 0.$$

$a_\alpha(x)$: indep. of u $\quad$ a_0: indep. of $D^k u$

- The PDE (*) is quasi linear if it has the form:

$$\sum_{|\alpha|\le k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0$$

- The PDE (*) is called fully-nonlinear if it depends nonlinearly upon the highest order derivatives $D^k u$.

Examples

The list of many specific PDE of interest in current research.

a. Linear

- Laplace's eq. $\Delta u = 0$
- Helmholtz's eq. $\Delta u + \lambda u = 0$
- Linear transport eq. $u_t + \underbrace{\sum_{i=1}^n b^i u_{x_i}}_{} = 0$ or $u_t + \vec{b} \cdot Du = 0$
- Liouville's eq. $u_t + D(\vec{b} \cdot u) = 0$

- Liouville's eq. $u_t + D(\vec{b} \cdot u) = 0$ $\quad u_t + \dot{b} \cdot Du = 0$

☆ Heat eq. $u_t = \Delta u$

- Schrödinger's eq. $i\partial_t u = \Delta u$

- Kolmogorov's eq.

$$u_t = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} - \sum_{i=1}^{n} b^i u_{x_i}$$

- Fokker-Planck's eq.

$$u_t = \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i}$$

☆ Wave eq. $u_{tt} = \Delta u$

[General Wave eq. $u_{tt} = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} - \sum_{i=1}^{n} b^i u_{x_i}$]

- Airy's eq. $u_t + u_{xxx} = 0$

- Beam eq. $u_t + u_{xxxx} = 0$

b. Nonlinear

- Nonlinear Poisson eq. $\Delta u = f(u)$

- p-Laplacian eq. $\operatorname{div}\left(|Du|^{p-2} Du\right) = 0$

$$|Du| = \sqrt{\sum_{i=1}^{n} |u_{x_i}|^2}$$

$Du = \sqrt{\sum_{i=1}^{n} |u_{x_i}|^2}$

- Minimal-surface eq.

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+(Du)^2}}\right)=0$$

- Monge-Ampère eq.

$$\det(D^2u)=f(u)$$

- Hamilton-Jacobi eq.

$$u_t + \underset{\sim}{H}(Du, x)=0$$

- kdV eq. $u_t + uu_x + u_{xxx}=0$

C. linear and nonlinear systems.

- Maxwell's eq.s

$$\begin{cases} E_t = \operatorname{curl} B \\ B_t = -\operatorname{curl} E_t \\ \operatorname{curl} B = \operatorname{div} E = 0 \end{cases}$$

- Navier-Stokes eq.s

(for incompressive, viscous fluid)

$$\begin{cases} \vec{u}_t = \Delta\vec{u} - Dp - \vec{u}\cdot D\vec{u} \\ \operatorname{div}\vec{u}=0 \end{cases}$$

<u>Next time</u> : transport eq.